Affine Residues and Chamber Jumps for Hyperplane Arrangements

Brennan Dury

Abstract

We formulate Szenes' iterated residue as a chain map in the cohomology theory of meromorphic forms with poles in locally finite affine hyperplane arrangements. Although the chain map depends on a choice of complement of the target affine subspace, we show that after passing to cohomology the induced map is independent of this choice. Then we show that this residue restricts to the cohomology theory of rapidly decaying meromorphic forms with poles in the complexification of real locally finite affine hyperplane arrangements. In this setting, we derive a jump formula for contour integrals across faces.

1 Introduction

Let $\mathcal{P}$ be a locally finite real affine hyperplane arrangement in an affine space A with translation space V_A. If β is a meromorphic top-degree form with poles in the complexification of $\mathcal{P}$ and rapid decay in imaginary directions, $\int_{c+iV_A} \beta$ converges absolutely for all $c \in A$ outside the polar set. In fact, these integrals are locally constant in c. It is easy to show that crossing a wall changes the integral by a residue term. See Lemma 3.5. Theorem 3.6 generalizes this formula to crossing higher codimension faces.

We start with general complex affine hyperplane arrangements, where we formulate Szenes' iterated residue [5] as a chain map onto a flat of the arrangement. Szenes' construction assumes the affine hyperplanes intersect at a point, so we initially work in a complement of the flat. Although the chain map depends on this choice of complement, we show that after passing to cohomology, the resulting map is independent of this choice.

We then show that the residue, as well as the homotopy used to show independence of the complement, preserve rapid decay and thus induce a map on the cohomology of rapidly decaying meromorphic forms with poles in the complexification of $\mathcal{P}$. Finally, we prove the jump formula.

2 Residues along an Arrangement Flag

2.1 Normal fibers and flag tori

Let $\mathcal{A}$ be a finite affine hyperplane arrangement in a complex affine space X. A flat of $\mathcal{A}$ is a nonempty intersection of its hyperplanes. Write $\mathrm{lin}(Y)$ for the translation space of an

affine subspace Y, and $Z(q)$ for the zero set of an affine function q. For background on arrangements, see [4].

Definition 2.1. *A flag of $\mathcal{A}$ is a sequence $FL = (Y_r, \ldots, Y_0 = X)$ of flats with $Y_i \subsetneq Y_{i-1}$ and $\operatorname{codim}(Y_i) = i$. Its base is $Y = Y_r$.*

Fix such a flag with $r > 0$, and an open set $U \subseteq X$ meeting Y. Put $S = Y \cap U$ and $L = \operatorname{lin}(Y)$.

Definition 2.2. *An affine projection onto Y is an affine map $p : X \to Y$ whose restriction to Y is the identity. Its normal space $N = \ker(dp)$ is a complement to L, and its fibers are $p^{-1}(y) = y + N$.*

The hyperplanes of $\mathcal{A}$ containing Y induce the central arrangement

$$\mathcal{A}_N = \{N \cap \operatorname{lin}(D) : D \in \mathcal{A},\ Y \subseteq D\}$$

in the normal space, with the complete flag of flats

$$N = N_0 \supset N_1 \supset \cdots \supset N_r = 0, \qquad N_i = N \cap \operatorname{lin}(Y_i).$$

Choose a small torus $\Gamma \subset N \setminus \bigcup \mathcal{A}_N$ associated with this flag by the construction of Szenes [5, Section 3.1, equation (3.1)]. Orient Γ by ordering its positively oriented linking circles by flag steps $r, \ldots, 1$. We call the resulting oriented homology class the *flag class*, and Γ a *flag torus*.

We have an induced arrangement

$$\mathcal{A}_Y = \{D \cap Y : D \in \mathcal{A}, \quad \emptyset \neq D \cap Y \subsetneq Y\}$$

on the base. We say the regular locus is $S^\circ = S \setminus \bigcup \mathcal{A}_Y$. Near each $y \in S^\circ$, choose a neighborhood $W \subset S^\circ$ and a ball $B \subset N$ about zero such that $W + B \subset U$ meets no hyperplane in $\mathcal{A}$ not containing Y. Shrink Γ into B. Its translates over W then lie in $U \setminus |\mathcal{A}|$.

2.2 The flag residue

Write $\Omega^m_{\mathcal{A}}(U)$ for meromorphic m-forms on U with hyperplane poles in $\mathcal{A}$. We use the same notation for induced arrangements on affine subspaces. Write Ω^j_{hol} for holomorphic forms.

Addition parametrizes the family of contours over W:

$$\begin{aligned} H &: W \times \Gamma \longrightarrow U \setminus \bigcup \mathcal{A}, \quad H(y, n) = y + n, \\ \pi &: W \times \Gamma \longrightarrow W, \qquad\qquad \pi(y, n) = y. \end{aligned}$$

Write π_* for integration over the fibers Γ, with the convention $\pi_*(\gamma \wedge \theta) = (\int_\Gamma \theta)\gamma$ for $\gamma \in \Omega^a(W; \mathbb{C})$ and $\theta \in \Omega^r(\Gamma; \mathbb{C})$, where $\Omega^j(M; \mathbb{C})$ denotes the space of complex-valued smooth differential j-forms on M, and $r = \dim_{\mathbb{R}} \Gamma$. Since H is holomorphic in the base variable, $\pi_* H^* \beta$ is holomorphic on W.

Definition 2.3. *The residue associated with FL and p is the linear map*

$$\operatorname{Res}_{FL,p} : \Omega^m_{\mathcal{A}}(U) \longrightarrow \Omega^{m-r}_{\mathcal{A}_Y}(U \cap Y)$$

defined on every sufficiently small open set $W \subset S^\circ$ by

$$\operatorname{Res}_{FL,p} \beta|_W = \frac{1}{(2\pi i)^r} \pi_* H^* \beta \in \Omega^{m-r}_{\mathrm{hol}}(W), \tag{1}$$

for every sufficiently small flag torus Γ. Existence and uniqueness are established in Proposition 2.4.

Affine projections $p_i : Y_{i-1} \to Y_i$ are *compatible* with p if $p = p_r \circ \cdots \circ p_1$. Such projections always exist: split each $N_{i-1} = E_i \oplus N_i$, with E_i a complex line, and let p_i be projection along E_i. Conversely, a compatible factorization gives these splittings by $E_i = \ker(dp_i)$.

Proposition 2.4. *For every $\beta \in \Omega^m_{\mathcal{A}}(U)$, the integrals in (1) determine a unique meromorphic form on $U \cap Y$ with poles in $\mathcal{A}_Y$. Thus the flag residue is well-defined.*

For any compatible projections $p_i : Y_{i-1} \to Y_i$,

$$\operatorname{Res}_{FL,p} = \operatorname{Res}_{(Y,Y_{r-1}),p_r} \circ \cdots \circ \operatorname{Res}_{(Y_1,X),p_1}. \tag{2}$$

Proof. Fix $y \in S^\circ$. Choose a small ball $B \subset N$ about zero such that $y + B \subset U$ meets no hyperplane in $\mathcal{A}$ not containing Y. Small flag tori are homologous in $B \setminus \bigcup \mathcal{A}_N$, so Stokes shows that the residue is independent of the representative.

For compatible projections, choose positively oriented circles $\gamma_i \subset E_i$ of radii ϵ_i and take

$$\Gamma = \gamma_r \times \cdots \times \gamma_1, \qquad \epsilon_1 \ll \cdots \ll \epsilon_r.$$

Fubini expresses its integral as successive circle integrals over $\gamma_1, \ldots, \gamma_r$.

At each step, choose a transverse parameter t for the corresponding projection, with $D = \{t = 0\}$. A contributing term is $t^{-a}\eta(y,t) \wedge dt$, with η tangential, and Cauchy's formula gives, coefficientwise,

$$\frac{1}{2\pi i} \int_{|t|=\epsilon} \frac{\eta(y,t)}{t^a}\, dt = \frac{1}{(a-1)!} \partial_t^{a-1} \eta(y,0), \qquad a \geq 1. \tag{3}$$

Here η is holomorphic in t near zero away from the induced poles. Its coefficients have the form $\eta_I = f_I / \prod_j q_j^{b_j}$, with f_I holomorphic and q_j defining the other polar hyperplanes. Differentiation and restriction to $t = 0$ give poles only on $Z(q_j|_D)$. Iterating proves meromorphic extension with poles in $\mathcal{A}_Y$ and the composition formula (2). The local extensions agree and glue uniquely by the meromorphic identity principle [1, Chapter I]. □

Remark 2.5. *Let $\mathcal{B} \supseteq \mathcal{A}$ be a finite arrangement and $\beta \in \Omega^m_{\mathcal{A}}(U)$. Enlarging the arrangement to $\mathcal{B}$ does not change* $\operatorname{Res}_{FL,p} \beta$. *Indeed, near a point of $S \setminus \bigcup \mathcal{B}_Y$, choose a sufficiently small flag torus Γ for $\mathcal{B}_N$ by the nested-circle construction above. This torus avoids all hyperplanes in $\mathcal{A}_N \subseteq \mathcal{B}_N$, and the same construction makes it a representative of the flag class for $\mathcal{A}_N$. Using Γ to compute both residues gives identical integrals in (1). Their meromorphic extensions are therefore equal on $U \cap Y$ by the identity principle.*

2.3 Differential and projection homotopy

Write d_X and d_Y for the holomorphic de Rham differentials on X and Y, respectively.

Proposition 2.6. *The residue commutes with the de Rham differential:*

$$d_Y \operatorname{Res}_{FL,p} = \operatorname{Res}_{FL,p} d_X. \tag{4}$$

Proof. Fix a sufficiently small open set $W \subset S^\circ$. Let d_W and d denote the exterior differentials on W and $W \times \Gamma$, respectively. On holomorphic forms, d_W agrees with the restriction of d_Y.

Since Γ is compact and has no boundary, fiberwise Stokes [3, Lemma B.2(ii)] gives $d_W \pi_* = \pi_* d$.[1] Naturality of the exterior differential gives $dH^*\beta = H^*(d_X\beta)$. On W, (1) therefore gives

$$\begin{aligned} d_Y \operatorname{Res}_{FL,p} \beta &= \frac{1}{(2\pi i)^r} d_W \pi_* H^* \beta \\ &= \frac{1}{(2\pi i)^r} \pi_* dH^* \beta \\ &= \frac{1}{(2\pi i)^r} \pi_* H^*(d_X \beta) \\ &= \operatorname{Res}_{FL,p} d_X \beta. \end{aligned}$$

The meromorphic identity principle extends this equality to S, proving (4). □

Theorem 2.7 (Projection homotopy)**.** *For two affine projections $p_0, p_1 : X \to Y$, put*

$$p_s = (1-s)p_0 + s p_1, \qquad \xi(z) = p_0(z) - p_1(z) \in L.$$

Each p_s is a projection onto Y, and ξ is an affine vector field vanishing on Y. The degree $-r-1$ operator

$$\begin{aligned} K_{p_0,p_1} &: \Omega^m_{\mathcal{A}}(U) \longrightarrow \Omega^{m-r-1}_{\mathcal{A}_Y}(S), \\ K_{p_0,p_1}\beta &= \int_0^1 \operatorname{Res}_{FL,p_s}(\iota_\xi \beta)\, ds \end{aligned} \tag{5}$$

satisfies

$$\operatorname{Res}_{FL,p_1} - \operatorname{Res}_{FL,p_0} = d_Y K_{p_0,p_1} + K_{p_0,p_1} d_X. \tag{6}$$

Proof. Let $N_s = \ker(dp_s)$ for $0 \le s \le 1$. Set

$$\begin{aligned} &\phi_s : N_0 \longrightarrow N_s, && \phi_s(n) = n - s\, dp_1(n), \\ &H_s : Y \times N_0 \longrightarrow X, && H_s(y,n) = y + \phi_s(n). \end{aligned} \tag{7}$$

The map ϕ_s is an isomorphism, with $\phi_s^{-1}(v) = v - dp_0(v)$, and

$$p_s H_s(y,n) = y, \qquad \partial_s H_s(y,n) = -dp_1(n) = \xi(H_s(y,n)).$$

Since $dp_1(n) \in L \subseteq \operatorname{lin}(Y_i)$,

$$\phi_s\big(N_0 \cap \operatorname{lin}(Y_i)\big) = N_s \cap \operatorname{lin}(Y_i), \qquad 0 \le i \le r.$$

[1] The fiber-first convention in [3] differs from our base-first convention on forms of total degree m by the factor $(-1)^{r(m-r)}$. With our convention, their identity becomes $d_W\pi_* = \pi_* d$.

For $D = Z(q) \in \mathcal{A}$ with $Y \subseteq D$, $q|_Y = 0$ and $dq|_L = 0$, so

$$q(H_s(y,n)) = q(y) + dq(n) - s\,dq(dp_1(n)) = dq(n).$$

Thus ϕ_s carries a flag torus Γ in N_0 to a flag torus in N_s. Near each point of S°, compactness of $[0,1]$ allows us to choose an open set $W \subset S^\circ$ and a sufficiently small Γ such that

$$H_s(W \times \Gamma) \subset U \setminus \bigcup \mathcal{A}, \qquad 0 \le s \le 1.$$

Since $\partial\Gamma = 0$,

$$\partial([0,1] \times \Gamma) = (\{1\} \times \Gamma) - (\{0\} \times \Gamma).$$

On W, the residue formula (1) and the definition of K_{p_0,p_1} give

$$\begin{aligned} \mathrm{Res}_{FL,p_s}\,\beta &= \frac{1}{(2\pi i)^r}\pi_* H_s^*\beta, \\ K_{p_0,p_1}\beta &= \frac{1}{(2\pi i)^r}\int_0^1 \pi_* H_s^*(\iota_\xi\beta)\,ds. \end{aligned}$$

Using $\partial_s H_s = \xi \circ H_s$, fiberwise Stokes [3, Lemma B.2(ii)] on the cylinder therefore gives

$$\begin{aligned} \mathrm{Res}_{FL,p_1}\,\beta - \mathrm{Res}_{FL,p_0}\,\beta &= \frac{1}{(2\pi i)^r}\pi_*(H_1^*\beta - H_0^*\beta) \\ &= d_Y\left(\frac{1}{(2\pi i)^r}\int_0^1 \pi_* H_s^*(\iota_\xi\beta)\,ds\right) \\ &\quad + \frac{1}{(2\pi i)^r}\int_0^1 \pi_* H_s^*(\iota_\xi d_X\beta)\,ds \\ &= d_Y K_{p_0,p_1}\beta + K_{p_0,p_1}(d_X\beta). \end{aligned}$$

Since such open sets W cover S°, this proves (6) on S°.

To extend $K_{p_0,p_1}\beta$ across the hyperplanes, fix $y_0 \in S$ and choose a sufficiently small open set $V \subset S$ containing y_0. Cauchy's formula for a pole of order a uses $a-1$ derivatives. It therefore bounds the new pole orders in terms of the original pole orders and the arrangement, independently of the projection. Applying this successively along the flag, as in Proposition 2.4, gives a bound independent of s. Thus a single product D of powers of equations defining $\mathcal{A}_Y$ clears the poles for every s:

$$D\ \mathrm{Res}_{FL,p_s}(\iota_\xi\beta) \in \Omega^{m-r-1}_{\mathrm{hol}}(V), \qquad 0 \le s \le 1.$$

Successive applications of Cauchy's integral formula also give continuous dependence on s, uniformly on compact subsets of V. Thus the numerator in

$$\frac{1}{D}\int_0^1 D\ \mathrm{Res}_{FL,p_s}(\iota_\xi\beta)\,ds$$

is holomorphic on V, and this expression extends $K_{p_0,p_1}\beta$ meromorphically from $V \cap S^\circ$ to V, with poles only in $\mathcal{A}_Y$. Uniqueness of meromorphic extension then shows that $K_{p_0,p_1}\beta$ is meromorphic on S. $\square$

The construction extends to locally finite arrangements $\mathcal{A}$. Each point of $U \cap Y$ has a neighborhood in U meeting only finitely many hyperplanes. By Remark 2.5, the resulting local residue operators agree on overlaps. Formula (5) gives the same compatibility for the homotopy operators. They therefore glue to maps

$$\begin{aligned} \mathrm{Res}_{FL,p} &: \Omega^m_{\mathcal{A}}(U) \longrightarrow \Omega^{m-r}_{\mathcal{A}_Y}(U \cap Y), \\ K_{p_0,p_1} &: \Omega^m_{\mathcal{A}}(U) \longrightarrow \Omega^{m-r-1}_{\mathcal{A}_Y}(U \cap Y). \end{aligned}$$

The differential, homotopy, and composition identities remain valid, since they hold locally.

Write

$$H^m_{\mathcal{A}}(U) := \frac{\ker(d_X : \Omega^m_{\mathcal{A}}(U) \to \Omega^{m+1}_{\mathcal{A}}(U))}{d_X \Omega^{m-1}_{\mathcal{A}}(U)}$$

for meromorphic de Rham cohomology, with the same notation for induced arrangements. Proposition 2.6 shows that the residue preserves closed and exact forms, and Theorem 2.7 shows that its cohomology class is independent of the projection. We therefore obtain maps

$$\mathrm{Res}_{FL} : H^m_{\mathcal{A}}(U) \longrightarrow H^{m-r}_{\mathcal{A}_Y}(U \cap Y), \qquad [\beta] \longmapsto [\mathrm{Res}_{FL,p}\, \beta]. \tag{8}$$

By Proposition 2.4, these maps satisfy

$$\mathrm{Res}_{FL} = \mathrm{Res}_{(Y_r, Y_{r-1})} \circ \cdots \circ \mathrm{Res}_{(Y_1, X)},$$

where each successive residue is taken in the induced arrangement.

3 Rapid Decay and Chamber Jumps

3.1 Rapid decay cohomology and periods

Let A be an oriented k-dimensional real affine space with translation space V_A, and let $\mathcal{P}$ be a locally finite affine hyperplane arrangement. Fix a Euclidean norm on V_A and the induced norms on its complexification and exterior powers. Let

$$|\mathcal{P}| := \bigcup_{H \in \mathcal{P}} H.$$

The chambers are the connected components of $A \setminus |\mathcal{P}|$.

Write $A_{\mathbb{C}} = A + iV_A$ and $\mathcal{P}_{\mathbb{C}} = \{H_{\mathbb{C}} : H \in \mathcal{P}\}$. For real affine functions q, we identify q with its complexification. We write $Z_{\mathbb{C}}(q)$ for the complex zero set of q.

Definition 3.1. *Let $W \subseteq A$ be open and put $U = W + iV_A$. Write $\Omega^m_{\mathrm{rd}}(W, \mathcal{P})$ for the m-forms β meromorphic on U, with hyperplane poles in $\mathcal{P}_{\mathbb{C}}$, satisfying*

$$p_{K,\delta,N}(\beta) := \sup_{z \in E_A(K,\delta)} (1 + |\operatorname{Im} z|)^N \|\beta(z)\| < \infty, \qquad N \geq 0, \tag{9}$$

for every compact $K \subseteq W$ and every $\delta > 0$, where

$$E_A(K, \delta) := \{z \in A_{\mathbb{C}} : \operatorname{Re} z \in K, \quad \operatorname{dist}(z, |\mathcal{P}_{\mathbb{C}}|) \geq \delta\}.$$

We use $\operatorname{dist}(z,\emptyset)=\infty$ *and* $\sup\emptyset=0$ *for these seminorms. These seminorms define the topology on the space of forms. Set* $\Omega^m_{\mathrm{rd}}(W,\mathcal{P})=0$ *for* $m<0$*. Its cohomology is*

$$H^m_{\mathrm{rd}}(W,\mathcal{P}) := \frac{\ker(d:\Omega^m_{\mathrm{rd}}(W,\mathcal{P})\to\Omega^{m+1}_{\mathrm{rd}}(W,\mathcal{P}))}{d\Omega^{m-1}_{\mathrm{rd}}(W,\mathcal{P})}.$$

We give this cohomology the quotient topology.

Cauchy estimates on balls of fixed radius away from the poles show that the differential d is continuous on the decay complex.

Proposition 3.2. *For* $c\in W\setminus|\mathcal{P}|$*, integration defines a continuous linear functional*

$$J^A_c : H^k_{\mathrm{rd}}(W,\mathcal{P}) \longrightarrow \mathbb{C}, \qquad J^A_c([\beta]) := \frac{1}{(2\pi i)^k}\int_{c+iV_A}\beta. \tag{10}$$

The integral converges absolutely and, for each class $[\beta]$*, is constant on every connected component of* $W\setminus|\mathcal{P}|$*.*

Proof. Since the arrangement is real, $\operatorname{dist}(c+iV_A,|\mathcal{P}_{\mathbb{C}}|)\geq\operatorname{dist}(c,|\mathcal{P}|)>0$. A seminorm in (9) with $N>k$ therefore bounds the absolute integral by a constant times $\int_{V_A}(1+|t|)^{-N}\,dt<\infty$, proving convergence and continuity on top forms.

For two points whose joining segment K lies in $W\setminus|\mathcal{P}|$, apply Stokes' theorem to the closed top form β on the prism over iB_R, where $B_R\subset V_A$ is the ball of radius R. Its lateral boundary has volume $O(R^{k-1})$, and rapid decay uniformly over $K+iV_A$ bounds its integral by $O(R^{k-1-N})$. Letting $R\to\infty$ proves equality of the two periods. Thus the period is locally constant on $W\setminus|\mathcal{P}|$, hence constant on each connected component.

If $\beta=d\eta$ with $\eta\in\Omega^{k-1}_{\mathrm{rd}}(W,\mathcal{P})$, Stokes' theorem on $c+iB_R$ gives the same boundary estimate for η. Thus the integral of $d\eta$ is zero. □

When the value of $J^A_c([\beta])$ is independent of $c\in W\cap D$ for a chamber D meeting W, write $J^A_D([\beta])$ for the common value. This holds in particular when $W\cap D$ is connected.

3.2 Real flags and induced arrangements

For a face F of $\mathcal{P}$, put

$$L=\operatorname{aff}(F), \qquad V_L=\operatorname{lin}(L)=:\operatorname{lin}(F).$$

The induced arrangement

$$\mathcal{P}_L := \{D\cap L : D\in\mathcal{P},\quad \emptyset\neq D\cap L\subsetneq L\}, \tag{11}$$

with repetitions identified, is locally finite and has F as a chamber.

Let $FL=(F_r,\dots,F_0=\mathfrak{c})$ be a flag of faces with base $F=F_r$, where $F_i\subsetneq\overline{F_{i-1}}$, $\operatorname{codim}(F_i)=i$, and $r>0$. Complexifying the affine spans gives

$$FL_{\mathbb{C}}=(F_{r,\mathbb{C}},\dots,F_{0,\mathbb{C}}=A_{\mathbb{C}}), \qquad F_{i,\mathbb{C}}=\operatorname{aff}(F_i)+i\operatorname{lin}(F_i).$$

Write $\mathrm{Res}_{FL,p} := \mathrm{Res}_{FL_{\mathbb{C}},p}$, and use $\mathcal{P}_L$ in the definition of $E_L(K,\delta)$.

For a face G, put

$$\mathfrak{Q}(\mathcal{P}) := \{q : A \to \mathbb{R} \text{ affine} : Z(q) \in \mathcal{P}\},$$
$$\mathfrak{Q}(G,\mathcal{P}) := \{q \in \mathfrak{Q}(\mathcal{P}) : q|_G = 0\}.$$

For $q \in \mathfrak{Q}(\mathcal{P})$, write $\mathrm{sign}_G(q)$ for its constant sign on G. For $q \in \mathfrak{Q}(F,\mathcal{P})$, let $\mathrm{gi}(q)$ be the least $j \in \{1,\dots,r\}$ such that $q|_{F_j} = 0$.

Theorem 3.3 (Opposite flags [2, Opposite flags theorem])**.** *Let $\mathcal{P}$ be a locally finite affine hyperplane arrangement in A, and let $FL = (F_r,\dots,F_0 = \mathfrak{c})$ be a flag of faces with base $F = F_r$ of codimension $r > 0$ and top chamber $\mathfrak{c}$. For each $\alpha \in \{\pm 1\}^r$, there exists a chamber $\mathfrak{b}_{FL}(\alpha)$ and another flag of faces*

$$F = F_r(\alpha),\dots,F_0(\alpha) = \mathfrak{b}_{FL}(\alpha)$$

with signs

$$\mathrm{sign}_{F_i(\alpha)}(q) = \begin{cases} 0, & q \in \mathfrak{Q}(F_i,\mathcal{P}), \\ \alpha_{\mathrm{gi}(q)}\,\mathrm{sign}_{\mathfrak{c}}(q), & q \in \mathfrak{Q}(F,\mathcal{P}) \setminus \mathfrak{Q}(F_i,\mathcal{P}), \\ \mathrm{sign}_{\mathfrak{c}}(q), & q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F,\mathcal{P}). \end{cases}$$

We write $FL(\alpha)$ for this sign-indexed flag. In particular, $F_i(1^r) = F_i$ for all i, so $\mathfrak{b}_{FL}(1^r) = \mathfrak{c}$.

3.3 Residues on the decay complex

Proposition 3.4. *For every affine projection $p : A_{\mathbb{C}} \to L_{\mathbb{C}}$, $\mathrm{Res}_{FL,p}$ restricts to a continuous map*

$$\mathrm{Res}_{FL,p} : \Omega^m_{\mathrm{rd}}(W,\mathcal{P}) \longrightarrow \Omega^{m-r}_{\mathrm{rd}}(W \cap L, \mathcal{P}_L).$$

For any two such projections p_0, p_1, K_{p_0,p_1} restricts to a continuous map

$$K_{p_0,p_1} : \Omega^m_{\mathrm{rd}}(W,\mathcal{P}) \longrightarrow \Omega^{m-r-1}_{\mathrm{rd}}(W \cap L, \mathcal{P}_L).$$

Proof. Fix projections p_0, p_1, a compact $K \subset W \cap L$, and $\delta > 0$, and use H_s from (7). Let $\Gamma \subset \ker(dp_0)$ be a flag torus to be chosen below, and let π be projection onto the base. We use the integral formulas (1) and (5):

$$\mathrm{Res}_{FL,p_0}\,\beta = \frac{1}{(2\pi i)^r}\pi_* H_0^*\beta,$$
$$K_{p_0,p_1}\beta = \frac{1}{(2\pi i)^r}\int_0^1 \pi_* H_s^*(\iota_\xi \beta)\,ds.$$

To choose Γ uniformly on $E_L(K,\delta)$, choose a compact set $K' \subset W$ with $K \subset \operatorname{int} K'$. By local finiteness, only finitely many hyperplanes meet K', and the remaining hyperplanes have a common positive distance from K'. For each hyperplane meeting K', its defining equation satisfies

$$q|_L = 0 \implies q(H_s(y,n)) = dq(n),$$
$$q|_L \not\equiv 0 \implies \inf_{y \in E_L(K,\delta)} |q(y)| > 0.$$

We therefore choose Γ sufficiently small that, for some $\delta' > 0$,

$$H_s(E_L(K,\delta)\times\Gamma) \subset E_A(K',\delta') \qquad (0 \le s \le 1).$$

This choice of Γ makes both formulas valid throughout $E_L(K,\delta)$. Using the formula for H_s in (7), the integral formulas and the seminorm definition, together with compactness of $[0,1]\times\Gamma$, give

$$p_{K,\delta,M}(\mathrm{Res}_{FL,p_0}\,\beta) + p_{K,\delta,M}(K_{p_0,p_1}\beta) \le C_M p_{K',\delta',M}(\beta) \qquad (M \ge 0). \tag{12}$$

This proves decay and continuity. □

Proposition 2.6 and Theorem 2.7 now show that $\mathrm{Res}_{FL,p}$ descends to cohomology, giving a map

$$\mathrm{Res}_{FL} : H^m_{\mathrm{rd}}(W,\mathcal{P}) \longrightarrow H^{m-r}_{\mathrm{rd}}(W\cap L,\mathcal{P}_L), \qquad [\beta] \longmapsto [\mathrm{Res}_{FL,p}\,\beta] \tag{13}$$

independent of p. These maps commute with restriction of W and satisfy the composition formula (2).

3.4 The jump formula

An open set $W \subseteq A$ is *chamber-connected* if its intersection with each chamber is empty or connected.

For a face flag $FL = (F_r,\dots,F_0 = \mathfrak{c})$, put $V_i = \mathrm{lin}(F_i)$. Orient the line V_{i-1}/V_i so that the class of $x - y$ is positive for $x \in F_{i-1}$ and $y \in F_i$. Starting with the given orientation of $V_0 = V_A$, orient each V_i by

$$\det V_{i-1} = \det V_i \otimes \det(V_{i-1}/V_i) \qquad (1 \le i \le r),$$

so the quotient directions occur in the order $r,\dots,1$.

Lemma 3.5 (Codimension-one jump formula)**.** *Let $W \subseteq A$ be open and chamber-connected, and let C_+ and C_- be distinct chambers sharing a codimension-one face D, with $W \cap D \neq \emptyset$. Put $H = \mathrm{aff}(D)$, oriented by the face flag (D, C_+). For every $[\omega] \in H^k_{\mathrm{rd}}(W,\mathcal{P})$, the residue period is constant on $W \cap D$, and*

$$J^A_{C_+}([\omega]) - J^A_{C_-}([\omega]) = J^H_D(\mathrm{Res}_{(H_{\mathbb{C}},A_{\mathbb{C}})}[\omega]). \tag{14}$$

Proof. Fix $c \in W \cap D$ and a real transverse vector u pointing toward C_+. Choose $a > 0$ so that $K = c + [-a,a]u \subset W$ meets no hyperplane other than H and $c \pm au \in C_\pm$. Use projection onto $H_{\mathbb{C}}$ along $\mathbb{C}u$, and put

$$B = c + iV_H, \qquad Q_T = [-a,a] + i[-T,T], \qquad h(y,w) = y + wu.$$

Orient ∂Q_T counterclockwise. The only possible transverse pole inside Q_T is $w = 0$, and for $T \ge a$ the boundary contours stay a fixed positive distance from the poles. The one-variable residue theorem and Fubini give

$$J^H_c(\mathrm{Res}_{(H_{\mathbb{C}},A_{\mathbb{C}})}[\omega]) = \frac{1}{(2\pi i)^k}\int_{B\times\partial Q_T} h^*\omega.$$

Rapid decay makes the integrated horizontal edges vanish as $T \to \infty$. The right vertical side is traversed upward and the left downward, so chamber-connectedness gives

$$\lim_{T\to\infty} \frac{1}{(2\pi i)^k} \int_{B\times\partial Q_T} h^*\omega = J^A_{C_+}([\omega]) - J^A_{C_-}([\omega]).$$

Since $c \in W \cap D$ was arbitrary, the residue period is constant on $W \cap D$, proving (14). $\square$

Theorem 3.6 (Jump formula)**.** *Let $W \subseteq A$ be open and chamber-connected, and let $[\omega] \in H^k_{\mathrm{rd}}(W, \mathcal{P})$. Let $FL = (F_r, \ldots, F_0)$ have base $F = F_r$ of codimension $r > 0$, with $W \cap F \neq \emptyset$, and put $L = \mathrm{aff}(F)$. Then*

$$\boxed{\sum_{\alpha\in\{\pm1\}^r} \mathrm{sign}(\alpha) J^A_{F_0(\alpha)}([\omega]) = J^L_F(\mathrm{Res}_{FL}([\omega])),} \tag{15}$$

where $\mathrm{sign}(\alpha) = \prod_j \alpha_j$.

Proof. We argue by induction on r. For $r = 1$, the result is Lemma 3.5.

Let $r > 1$, and let $V \subset W$ be any convex open set meeting F. Put $H = \mathrm{aff}(F_1)$ and $FL' = (F_r, \ldots, F_1)$. Orient H by (F_1, F_0); the shorter flag FL' then induces on L the same orientation as FL. For each $\sigma \in \{\pm1\}^{r-1}$, the sign formula in Theorem 3.3 at $i = 1$ gives

$$F_1(+,\sigma) = F_1(-,\sigma) = \mathfrak{b}_{FL'}(\sigma).$$

The chambers $F_0(+,\sigma)$ and $F_0(-,\sigma)$ share this face, and the former lies on the same side of H as F_0. Lemma 3.5 therefore applies with this orientation of H and gives

$$J^A_{F_0(+,\sigma)}([\omega]) - J^A_{F_0(-,\sigma)}([\omega]) = J^H_{F_1(+,\sigma)}(\mathrm{Res}_{(H_{\mathbb{C}},A_{\mathbb{C}})}[\omega]).$$

The residue preserves rapid decay by Proposition 3.4. Restricting to the convex set $V \cap H$ and applying induction to FL', with periods on L computed in $V \cap L$, gives

$$\begin{aligned}\sum_{\alpha\in\{\pm1\}^r} \mathrm{sign}(\alpha) J^A_{F_0(\alpha)}([\omega]) &= \sum_{\sigma\in\{\pm1\}^{r-1}} \mathrm{sign}(\sigma) J^H_{F_1(+,\sigma)}(\mathrm{Res}_{(H_{\mathbb{C}},A_{\mathbb{C}})}[\omega]) \\ &= J^L_F(\mathrm{Res}_{FL'} \mathrm{Res}_{(H_{\mathbb{C}},A_{\mathbb{C}})}[\omega]) \\ &= J^L_F(\mathrm{Res}_{FL}[\omega]).\end{aligned}$$

The last equality is (2), using compatible projections. The left side is independent of V. Since V is arbitrary, the residue period has this same value throughout $W \cap F$. $\square$

References

[1] Henri Cartan. *Elementary Theory of Analytic Functions of One or Several Complex Variables*. Dover Publications, 1995.

[2] Brennan Dury. Quotient lexicographic traversal of affine hyperplane chambers. Preprint, 2026. `https://github.com/BrennanDury/MultivariateZeta/blob/b41091b8947a7ff3df1a840ce364ca9a9bea98d0/Graph/Graph.tex`.

[3] Yi Lin and Reyer Sjamaar. A Thom isomorphism in foliated de Rham theory. *Indagationes Mathematicae*, 32(1):121–150, 2021. https://arxiv.org/abs/2001.11848.

[4] Peter Orlik and Hiroaki Terao. *Arrangements of Hyperplanes*. Grundlehren der mathematischen Wissenschaften 300. Springer, 1992.

[5] András Szenes. Iterated residues and multiple Bernoulli polynomials. *International Mathematics Research Notices*, 1998(18):937–956, 1998. https://arxiv.org/abs/hep-th/9707114.