Mellin Asymptotics under Anisotropic Dilations

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Abstract

Classical one-dimensional Mellin asymptotics relate poles of the Mellin transform to asymptotic terms that are exponentials times polynomials in the logarithmic scale. We develop a multivariate counterpart for vector-space valued tempered distributions on the closed quadrant under anisotropic dilations, using chamber traversal and residue jump formulas for affine hyperplane arrangements. Under appropriate assumptions to support Mellin inversion, we construct finite residue expansions with uniform remainder bounds. For each fixed input recovered by Mellin inversion, finitely many residue formulas capture every bounded normalized dilation net with diverging parameters, after passage to a subnet. The resulting distribution-valued polynomials and exponential factors determine the normalization scales and leading distributions. We also establish two-sided bounds on the polynomials and homogeneity of any nonzero limiting distribution under dilation in the limiting direction.

1 Introduction

Let $Q = [0, \infty)^d$. Fix a complex topological vector space $\mathcal{F}$ and write $\mathcal{F}'_\beta$ for its continuous dual with the topology of uniform convergence on bounded subsets of $\mathcal{F}$. Let $\mathcal{S}'_Q(\mathcal{F}'_\beta)$ be the set of $\mathcal{F}'_\beta$-valued tempered distributions $\mathcal{S}(\mathbb{R}^d) \to \mathcal{F}'_\beta$ with support in Q, equipped with the topology of uniform convergence on bounded sets of Schwartz tests and bounded sets of $\mathcal{F}$-tests. View $W = \mathbb{R}^d$ as the space of logarithmic scales and $A = W^*$ as the Mellin exponent space. We study anisotropic dilation asymptotics of $U \in \mathcal{S}'_Q(\mathcal{F}'_\beta)$ through Mellin inversion. For $\omega \in W$, define

$$D_\omega(x) = (e^{-\omega_1} x_1, \dots, e^{-\omega_d} x_d), \qquad T_\omega U = (D_\omega)_* U.$$

We assume that the Mellin transform of U extends meromorphically to $A_{\mathbb{C}}$, has poles on the complexification of a locally finite real affine hyperplane arrangement in A with finitely many normal directions, and has polynomial vertical growth away from the poles, with continuous dependence on the auxiliary test (Section 4). For a fixed polar arrangement and initial chamber, let $\mathcal{A} \subseteq \mathcal{S}'_Q(\mathcal{F}'_\beta)$ be the space of distributions satisfying these hypotheses. We construct a projection $E : \mathcal{A} \to \mathcal{A}$ by a cutoff limit. Its kernel consists precisely of the distributions supported on ∂Q. We prove that Mellin transformation followed by inversion equals E. Consequently, Mellin inversion alone cannot in general determine limiting distributions of $T_{\omega_i} U$, since it does not detect the boundary component $U - EU$. We therefore seek asymptotic forms in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$ along nets $(\omega_i)_{i \in I}$:

$$T_{\omega_i} EU = a_i (Z + o_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(1)), \qquad \|\omega_i\| \to \infty, \quad a_i > 0, \quad Z \neq 0, \tag{1}$$

with the family $\{T_{\omega_i} EU / a_i : i \in I\}$ bounded in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$. We will assume this form exists and derive a formula for Z as a sum of dilated residue distributions. Additionally, we will derive the rate of a_i, also determined by the residues.

We use the residue jump formula of [3] to shift the Mellin inversion contour across poles, whose residues contribute the dominant terms. Each jump requires a choice of a face to cross and a flag of faces based there. We organize these choices using the traversal strategy of [2], which places the remainder contours in subsequent chambers beyond any prescribed decay threshold.

In Section 6 we construct linear maps $\mathcal{L}_{S,C,D}$ and $\mathcal{R}_{S,C,D}$, indexed by affine geometric data, with $\mathcal{L}_{S,C,D} + \mathcal{R}_{S,C,D} = E$. The first collects the dominant residue terms and the second the remainder integrals. Theorem 6.15 selects a finite family for each U and proves that, along suitable subnets, the remainders are $o_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(a_i)$.

Theorem 7.1 then gives, after passage to a subnet,

$$T_{\omega_i} EU = \sum_{\alpha \in F_{S,C}} e^{-\alpha(Y_i)} T_b P_{S,C,\alpha}(Y_i) + o_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(a_i).$$

Here $\omega_i = Y_i + b_i$, the b_i converge to b, the α are real linear forms on S, and the $P_{S,C,\alpha}$ are polynomials with values in finite-dimensional spaces of residue distributions. Fixing a norm on each of these spaces, define

$$M_{S,C}(Y) = \max_{\alpha \in F_{S,C}} e^{-\alpha(Y)} \|P_{S,C,\alpha}(Y)\|.$$

Consequently, Corollary 7.2 shows that Z is a finite sum of dilated residue profiles and

$$a_i \sim c\, M_{S,C}(Y_i), \qquad c > 0.$$

Each character is determined by a linear program over a chamber of the polar affine hyperplane arrangement. The polynomial blocks come from residue derivatives at the maximizing faces. Their degrees are bounded in terms of the polar divisor of the Mellin transform of U.

Section 8 additionally shows that if the polynomial $P_{S,C,\alpha}$ is not constant, there exist $c, C > 0$ such that

$$c(1 + \rho(Y))^{1/D} \le \|P_{S,C,\alpha}(Y)\| \le C(1 + \rho(Y))^D, \qquad Y \in S,$$

where $D \ge 1$ is the degree of $P_{S,C,\alpha}$ and $\rho(Y)$ is the distance from Y to the subspace of translations leaving this polynomial unchanged. Additionally, we show that if $\omega_i/\|\omega_i\| \to \theta$ in (1), then there is $\lambda \in \mathbb{R}$ such that

$$T_{h\theta} Z = e^{-h\lambda} Z \quad (h \in \mathbb{R}), \qquad \frac{\log a_i}{\|\omega_i\|} \longrightarrow -\lambda.$$

2 Chamber jumps and decay cohomology

We first formulate contour motion independently of a particular Mellin transform. A vertical contour is specified by its real base point, as in Mellin inversion. Moving that point within a chamber leaves the integral unchanged, while crossing its boundary produces residues.

2.1 Cohomology and periods

Let A be an oriented real affine space of dimension d, with translation space V_A, and let $\mathcal{P}$ be a locally finite real affine hyperplane arrangement. Local finiteness means that every compact subset of A meets only finitely many hyperplanes. Write $|\mathcal{P}| = \bigcup_{H \in \mathcal{P}} H$. The chambers are the connected components of $A \setminus |\mathcal{P}|$; more generally, faces are the nonempty sets obtained by prescribing the signs, including zero, of defining affine forms for all hyperplanes. A face is open in its affine hull.

We write $\mathrm{aff}(F)$ for the smallest affine subspace containing F, and $\dim F$ for the dimension of that subspace. For the traversal we assume that the hyperplanes have only finitely many normal directions, although there may be infinitely many parallel hyperplanes.

Write $A_{\mathbb{C}} = A + iV_A$ for the complexification of A, and $\mathcal{P}_{\mathbb{C}} = \{H_{\mathbb{C}} : H \in \mathcal{P}\}$ for the arrangement of complexified hyperplanes. Its union is $|\mathcal{P}_{\mathbb{C}}| = \bigcup_{H\in\mathcal{P}} H_{\mathbb{C}}$.

We use the rapid decay cohomology of [3, "Rapid decay cohomology and periods"].

Definition 2.1 (Rapid decay cohomology)**.** Write $\Omega^m_{\mathrm{rd}}(A,\mathcal{P})$ for meromorphic m-forms on $A_{\mathbb{C}}$, with poles in $|\mathcal{P}_{\mathbb{C}}|$, whose coefficients satisfy

$$\sup_{z\in E}(1+\|\Im z\|)^N|b(z)| < \infty \qquad \text{for every } N \geq 0 \tag{2}$$

whenever E has compact real part and positive distance from the complex polar arrangement. We call this rapid vertical decay. We equip $\Omega^m_{\mathrm{rd}}(A,\mathcal{P})$ with the topology defined by these seminorms on its coefficients. The differential preserves this space by [3, "Rapid decay cohomology and periods"]:

$$d : \Omega^m_{\mathrm{rd}}(A,\mathcal{P}) \longrightarrow \Omega^{m+1}_{\mathrm{rd}}(A,\mathcal{P}).$$

Define

$$H^m_{\mathrm{rd}}(A,\mathcal{P}) = \frac{\{\beta \in \Omega^m_{\mathrm{rd}}(A,\mathcal{P}) : d\beta = 0\}}{d\Omega^{m-1}_{\mathrm{rd}}(A,\mathcal{P})}.$$

Here $\Omega^k_{\mathrm{rd}}(A,\mathcal{P}) = 0$ for $k < 0$.

For a face F of $\mathcal{P}$, let $L = \mathrm{aff}(F)$, with translation space V_L. The induced arrangement on L is

$$\mathcal{P}_L = \{H \cap L : H \in \mathcal{P}, \quad \emptyset \neq H \cap L \subsetneq L\}, \tag{3}$$

with repetitions identified. The face F is a chamber of this arrangement. Given an orientation on L, let

$$J_F : H^{\dim F}_{\mathrm{rd}}(L,\mathcal{P}_L) \longrightarrow \mathbb{C}$$

be defined by

$$J_F([\beta]) = \frac{1}{(2\pi i)^{\dim F}} \int_{c+iV_L} \beta, \qquad c \in F. \tag{4}$$

This is absolutely convergent and independent of c and the choice of representative β by [3, "Rapid decay cohomology and periods"]. We call this contour integral a period. In dimension zero it is evaluation at the point F, with the chosen orientation.

2.2 The chamber graph and the jump formula

Fix a class $u \in H^d_{\mathrm{rd}}(A,\mathcal{P})$ and a nonconstant real affine objective ω on A. The linear program on a chamber $\mathfrak{c}$ has value

$$\lambda_A(\mathfrak{c},\omega) = \sup_{x\in\overline{\mathfrak{c}}} \omega(x). \tag{5}$$

Here and below "LP value" means the value of this linear program. A larger value allows the real contour to move farther in the direction of increasing ω. For a fixed representative β of u, this improves the exponential decay of $J_{\mathfrak{c}}([e^{-t\omega}\beta])$ as $t \to \infty$.

A face flag is a chain $F_r, \dots, F_0$ with F_0 a chamber, $F_i \subset \overline{F_{i-1}}$, and $\dim F_i = d - i$. Its base is F_r and its top is F_0. For a flag $FL = (F_r, \dots, F_0)$, put $F = F_r$ and $L = \mathrm{aff}(F)$. The flag residue of [3, "Residues on the decay complex"] gives a map

$$\mathrm{Res}_{FL} : H^m_{\mathrm{rd}}(A, \mathcal{P}) \longrightarrow H^{m-r}_{\mathrm{rd}}(L, \mathcal{P}_L).$$

The flag induces an orientation on L from that on A: writing $V_i = \mathrm{lin}(F_i)$, use $\det V_{i-1} = \det V_i \otimes \det(V_{i-1}/V_i)$ successively, with the quotient line pointing toward F_{i-1}. Write $J_{F,FL}$ for the period with this induced orientation.

Choose the flag selection strategy Φ supplied by [2, "Progress and escape for the greedy strategy"], a function

$$\Phi : \{\text{chambers of } \mathcal{P}\} \longrightarrow \{\text{face flags of } \mathcal{P}\}$$

assigning to each chamber $\mathfrak{c}$ a flag $FL = \Phi(\mathfrak{c})$ with top $\mathfrak{c}$. The directed graph G_Φ has the chambers of $\mathcal{P}$ as vertices. If the selected flag FL has base F of codimension $r > 0$, the target-chamber construction in [2] gives chambers $\mathfrak{b}_{FL}(\alpha)$, indexed by $\alpha \in \{\pm 1\}^r$, with $\mathfrak{b}_{FL}(1^r) = \mathfrak{c}$. To describe them, write the flag as $F_r = F, \dots, F_0 = \mathfrak{c}$. The sign $\alpha_i = -1$ reverses the side of every hyperplane containing F_i but not F_{i-1}; $\alpha_i = 1$ preserves it. All other hyperplane signs agree with those of $\mathfrak{c}$. The outgoing edges at $\mathfrak{c}$ are

$$\mathfrak{c} \xrightarrow{\alpha} \mathfrak{b}_{FL}(\alpha), \qquad \alpha \in \{\pm 1\}^r \setminus \{1^r\}.$$

A chamber of infinite LP value has the codimension-zero flag $\Phi(\mathfrak{c}) = (\mathfrak{c})$ and no outgoing edges.

At a chamber $\mathfrak{c}$ of finite LP value, the base of $FL = \Phi(\mathfrak{c})$ is the optimal face $F = F(\mathfrak{c}, \omega)$:

$$\overline{F} = \{x \in \overline{\mathfrak{c}} : \omega(x) = \lambda_A(\mathfrak{c}, \omega)\}.$$

Write $\mathrm{sgn}(\alpha) = \prod_j \alpha_j$. Here $1^r = (1, \dots, 1)$ denotes the sign choice that leaves the chamber unchanged. The jump formula from [3] is

$$\boxed{J_{\mathfrak{c}}(u) = J_{F,FL}(\mathrm{Res}_{FL}\, u) - \sum_{\alpha \neq 1^r} \mathrm{sgn}(\alpha) J_{\mathfrak{b}_{FL}(\alpha)}(u).} \tag{6}$$

Observe that the sum has one chamber term for each outgoing edge, with coefficient $-\mathrm{sgn}(\alpha)$. Consider applying the jump formula to each new chamber $\mathfrak{b}_{FL}(\alpha)$ using the flag $\Phi(\mathfrak{b}_{FL}(\alpha))$ and substituting the result into the current formula for $J_{\mathfrak{c}}(u)$. Successive substitutions then follow directed paths in G_Φ, with coefficients given by products of the edge coefficients $-\mathrm{sgn}(\alpha)$.

3 Finite expansions

Using the graph and jump formula of Section 2.2, we seek an expansion whose remainder chambers satisfy $\lambda_A(\mathfrak{c}, \omega) > T$, for a prescribed threshold T. Such a chamber contains a point c with $\omega(c) > T$. Indeed, a point of its closure above T can be approximated by interior points. Section 4 uses this LP bound to control remainders in asymptotic expansions of dilated distributions.

Theorem 3.1 (Expansion beyond a threshold)**.** Suppose $\mathcal{P}$ is locally finite with finitely many normal directions, and ω is nonconstant. For every chamber $\mathfrak{c}_0$ and threshold $T \in \mathbb{R}$, there are finitely many flags FL_p with base faces F_p, chambers $\mathfrak{c}_\ell$, and integer coefficients a_p, a_ℓ such that, writing $L_p = \operatorname{aff}(F_p)$, every $u \in H^d_{\mathrm{rd}}(A, \mathcal{P})$ satisfies

$$\begin{aligned} J_{\mathfrak{c}_0}(u) = &\sum_p a_p J_{F_p, FL_p}(\operatorname{Res}_{FL_p} u) \\ &+ \sum_\ell a_\ell J_{\mathfrak{c}_\ell}(u). \end{aligned} \tag{7}$$

Each $\mathfrak{c}_\ell$ satisfies $\lambda_A(\mathfrak{c}_\ell, \omega) > T$. Every L_p is a proper affine subspace of A, and ω is constant on it with value at most T.

Proof. Starting with $J_{\mathfrak{c}_0}(u)$, apply (6) at each chamber $\mathfrak{c}$ of LP value at most T, using its selected flag $\Phi(\mathfrak{c})$. Keep its residue term and repeat on the new chamber terms, stopping each branch when its LP value exceeds T.

If the initial LP value exceeds T, no substitution is needed. Otherwise the uniform escape theorem of [2] bounds the length of every branch before it reaches such a chamber. Each vertex has at most $2^d - 1$ outgoing edges, so the expansion is finite.

Substitution gives (7), with the coefficient along a path labelled $\alpha_1, \dots, \alpha_j$ equal to

$$a = \prod_{i=1}^{j} (-\operatorname{sgn}(\alpha_i)), \qquad a = 1 \text{ for the empty path.} \tag{8}$$

The stopping rule gives the remainder bound. Each F_p is an optimal face at a processed chamber, so $\omega|_{L_p}$ is constant with value at most T; L_p is proper because ω is nonconstant. □

Each residue term is a chamber period on a smaller affine space, with the orientation inherited from its flag, so Theorem 3.1 can be applied to it again.

3.1 Weights on a face

Let $W \subseteq \operatorname{Aff}(A, \mathbb{R})$ be a vector space of real affine weights whose linear parts span V_A^*, and fix a norm on W. The contour improvement on an affine space L depends on the linear part of the restricted weight. To measure it, put

$$\begin{aligned} &W_L = \{X \in W : X|_L \text{ is constant}\}, \\ &\pi_L : W \longrightarrow W/W_L, \qquad s_L(\omega) = \|\pi_L \omega\|. \end{aligned} \tag{9}$$

where the norm on W/W_L is the quotient norm. We use these definitions for $L = \operatorname{aff}(F)$, including $L = A$, and write $W_F = W_{\operatorname{aff}(F)}$, $\pi_F = \pi_{\operatorname{aff}(F)}$, and $s_F = s_{\operatorname{aff}(F)}$. Thus $s_L(\omega) = 0$ precisely when ω is constant on L. When ω is constant on L, the factor $e^{-\omega|_L}$ multiplies an integrand on L by one scalar and provides no direction in which to move its contour. The quotient W/W_L keeps exactly the part that can improve its exponent. If $X \in W_F$, let $\ell_F(X)$ be its constant value on F.

4 Mellin representation

Mellin inversion supplies the contour integral to which the geometric expansions of Section 3 will be applied. The principal point is to express every test pairing using the same polar arrangement, with contour bounds controlled by seminorms of the test function.

4.1 Dilations and the Mellin transform

Put $Q = [0,\infty)^d$, $X = (0,\infty)^d$, and $\mathcal{H} = \mathcal{S}(\mathbb{R}^d)$, with $d \geq 1$. Fix a complex topological vector space $\mathcal{F}$, and equip its continuous dual $\mathcal{F}'_\beta$ with uniform convergence on bounded sets. Write

$$\mathcal{S}'_Q(\mathcal{F}'_\beta) = \{U \in \mathcal{L}(\mathcal{H}, \mathcal{F}'_\beta) : \operatorname{supp} U \subseteq Q\}.$$

We use $\langle U, h\rangle = U(h) \in \mathcal{F}'$ and the scalar pairing $\langle U(h), f\rangle$. Equip $\mathcal{S}'_Q(\mathcal{F}'_\beta)$ with the topology of uniform convergence on bounded sets of both tests, defined by the seminorms

$$\|V\|_{\mathcal{B}} = \sup_{\substack{h\in A_0\\ f\in B}} |\langle V(h), f\rangle|, \qquad \mathcal{B} = A_0 \times B, \quad A_0 \subset \mathcal{H},\ B \subset \mathcal{F} \text{ bounded.} \tag{10}$$

For a positive scalar function a, we write $R = o_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(a)$ when $R/a \to 0$ in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$ and $R = O_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(a)$ when R/a is bounded.

Pairings with $U \in \mathcal{S}'_Q(\mathcal{F}'_\beta)$ depend only on $h|_Q$. We will also use seminorms measured on Q,

$$p_{N,\alpha}(h) = \sup_{x\in Q}(1 + \|x\|)^N |\partial^\alpha h(x)|.$$

Continuous linear extension of Schwartz functions from Q to $\mathbb{R}^d$ allows continuous functionals supported in Q to be bounded using finitely many of these seminorms.

Let $W = \mathbb{R}^d$ be the logarithmic scale space and $A = W^*$ the Mellin exponent space, with the dual coordinate bases. We use their Euclidean norms and give A its coordinate orientation. The pairing identifies W with the linear weights on A: $\omega(z) = \langle z, \omega\rangle$, extended complex linearly in z. For $\omega \in W$, define

$$D_\omega : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \qquad D_\omega(x) = (e^{-\omega_1}x_1, \dots, e^{-\omega_d}x_d). \tag{11}$$

Since $D_\omega(Q) = Q$, pushforward defines

$$T_\omega : \mathcal{S}'_Q(\mathcal{F}'_\beta) \longrightarrow \mathcal{S}'_Q(\mathcal{F}'_\beta), \qquad T_\omega U = (D_\omega)_* U. \tag{12}$$

Thus

$$\langle T_\omega U, h\rangle = \langle U, h \circ D_\omega\rangle, \qquad T_{\omega+\eta} = T_\omega T_\eta.$$

For bounded ω, the maps $h \mapsto h \circ D_\omega$ send bounded Schwartz sets to bounded Schwartz sets, and their dependence on ω is continuous uniformly on such sets. Consequently dilations act continuously on $\mathcal{S}'_Q(\mathcal{F}'_\beta)$, locally equicontinuously in ω.

Mellin transformation is Fourier transform in logarithmic coordinates. Put

$$\ell : X \longrightarrow W, \qquad \ell(x) = (\log x_1, \dots, \log x_d), \qquad v = \ell_*(U|_X).$$

Restriction gives an $\mathcal{F}'_\beta$-valued distribution on X, and ℓ_* is the pushforward $\langle v, \phi\rangle = \langle U|_X, \phi \circ \ell\rangle$ for compactly supported smooth ϕ on W. For $z \in A_{\mathbb{C}}$, write

$$x^z = \exp\langle z, \ell(x)\rangle = \prod_i \exp(z_i \log x_i).$$

Since $\ell(D_\omega x) = \ell(x) - \omega$, $(D_\omega x)^z = e^{-\omega(z)} x^z$.

We use the Fourier convention

$$\widehat{g}(\xi) = \int_W e^{-i\langle \xi, y\rangle} g(y)\, dy, \qquad g \in \mathcal{S}(W), \quad \xi \in A.$$

The Fourier transform maps $\mathcal{S}(W)$ to $\mathcal{S}(A)$ and extends by duality to $\mathcal{F}'_\beta$-valued tempered distributions. For $c \in A$ such that

$$w_c = e^{\langle c,y\rangle} v \in \mathcal{L}(\mathcal{S}(W), \mathcal{F}'_\beta),$$

define the Mellin transform $M_U(c + i\cdot) \in \mathcal{L}(\mathcal{S}(A), \mathcal{F}'_\beta)$ by

$$M_U(c + i\cdot)[\varphi] = w_c(\widehat{\varphi}(-\cdot)), \qquad \varphi \in \mathcal{S}(A). \tag{13}$$

With $d^\times x = \prod_i dx_i/x_i$, this convention gives, for a scalar distribution $U = u(x)d^\times x$,

$$M_U(z) = \int_X u(x)x^z \, d^\times x$$

whenever the integral converges absolutely.

Fix a locally finite real affine hyperplane arrangement $\mathcal{P}$ in A with finitely many normal directions, containing

$$\mathcal{P}_0 = \big\{\{c : c_i = n\} : 1 \le i \le d, \quad n \in \mathbb{Z}_{\ge 0}\big\}.$$

These coordinate hyperplanes contain the poles of the test kernels in Lemma 4.3. Fix an initial chamber $\mathfrak{c}_0 \subset (-\infty, 0)^d$ of $\mathcal{P}$.

We require w_c to be tempered for every $c \in \mathfrak{c}_0$. Since $\mathfrak{c}_0$ is open, the transforms in (13) then have a holomorphic $\mathcal{F}'_\beta$-valued representative on $\mathfrak{c}_0 + iA$. The case $\mathcal{F}'_\beta = \mathcal{F} = \mathbb{C}$ is classical [4, Lemma 3.2], and we generalize the proof in Appendix B.1. We assume this representative meromorphically continues to $A_\mathbb{C} \cong \mathbb{C}^d$ and satisfies the following conditions.

(i) There are integers $m_H \ge 0$, $H \in \mathcal{P}$, independent of f, which bound the poles of every $\langle M_U(\cdot), f\rangle$. Locally choose a product D of defining affine forms with these multiplicities, clearing all poles in an open set $V \subset A_\mathbb{C}$. The map

$$\mathcal{F} \longrightarrow \mathcal{O}(V), \qquad f \longmapsto D(\cdot)\langle M_U(\cdot), f\rangle$$

is continuous for the compact-open topology.

(ii) On every $\Omega \subset A_\mathbb{C}$ with compact real part and positive distance from $|\mathcal{P}_\mathbb{C}|$, there is $N_\Omega \ge 0$ for which

$$q_\Omega(f) := \sup_{z\in\Omega} (1 + \|\Im z\|)^{-N_\Omega} |\langle M_U(z), f\rangle| \tag{14}$$

is a finite continuous seminorm on $\mathcal{F}$.

These are the Mellin hypotheses. They define the vector space

$$\mathcal{A}(\mathcal{P}, \mathfrak{c}_0) = \{U \in \mathcal{S}'_Q(\mathcal{F}'_\beta) : U \text{ satisfies the Mellin hypotheses}\}. \tag{15}$$

Direct computation shows

$$M_{T_\omega U}(z) = e^{-\omega(z)} M_U(z).$$

Thus $\mathcal{A}(\mathcal{P}, \mathfrak{c}_0)$ is an invariant subspace of every dilation, since this exponential has bounded modulus on compact real parts.

4.2 What Mellin inversion recovers

The Mellin transform uses only $U|_X$: any distribution supported on ∂Q has zero interior Mellin transform. We must therefore specify which extension of $U|_X$ to the closed quadrant inversion produces. We will show in Proposition 4.5 that it is the cutoff extension defined below.

A cutoff is a function $\chi \in C^\infty(\mathbb{R}^d)$ that vanishes whenever any coordinate is less than a fixed positive number and equals 1 whenever all coordinates are sufficiently large. We require all its Euler derivatives $\prod_i (x_i\partial_{x_i})^{\alpha_i}\chi$ to be bounded. Products of one-variable cutoffs have these properties. For each cutoff, set $\chi_n(x) = \chi(nx)$ for integers $n \geq 1$. Multiplication by χ_n maps $\mathcal{H}$ to $\mathcal{H}$ and, by duality, $\mathcal{S}'_Q(\mathcal{F}'_\beta)$ to $\mathcal{S}'_Q(\mathcal{F}'_\beta)$. Define $\operatorname{dom} E$ to consist of the $U \in \mathcal{S}'_Q(\mathcal{F}'_\beta)$ for which $\chi_n U$ has a limit in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$ independent of the cutoff, and denote that limit by EU. Thus

$$\langle EU, h\rangle = \lim_{n\to\infty} \langle U, \chi_n h\rangle, \tag{16}$$

with convergence uniform on the bounded test sets in (10).

Proposition 4.1. For every $U \in \operatorname{dom} E$, (16) defines $EU \in \mathcal{S}'_Q(\mathcal{F}'_\beta)$, $(EU)|_X = U|_X$, and $EU \in \operatorname{dom} E$ with $E(EU) = EU$. Moreover

$$\ker E = \{W \in \operatorname{dom} E : \operatorname{supp} W \subseteq \partial Q\}.$$

For each cutoff χ, the sequence $\chi_n U$ converges to EU in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$.

Proposition 4.2. Suppose that, for some $c \in (-\infty, 0)^d$,

$$w_c = e^{\langle c,y\rangle} v \in \mathcal{L}(\mathcal{S}(W), \mathcal{F}'_\beta). \tag{17}$$

Then $U \in \operatorname{dom} E$, with

$$\langle EU, h\rangle = \langle w_c, e^{-\langle c,y\rangle} h(e^y)\rangle \qquad (h \in \mathcal{H}). \tag{18}$$

The proofs are in Appendix B.2. In particular, $\mathcal{A}(\mathcal{P}, \mathfrak{c}_0) \subseteq \operatorname{dom} E$. By Proposition 4.1, EU has the same interior restriction as U, so this input space is preserved by E: its restriction is $E : \mathcal{A} \to \mathcal{A}$, and $M_{EU} = M_U$.

In the scalar case, for a positive tempered measure giving no mass to ∂Q, dominated convergence gives $EU = U$. Consequently, Mellin inversion can recover such positive tempered measures exactly when the Mellin hypotheses hold.

4.3 From Mellin inversion to chamber periods

To recover the pairing of EU with $h \in \mathcal{H}$, we transform h with the exponent opposite to the one in M_U:

$$H_h(z) = \int_X x^{-z} h(x)\, d^\times x, \qquad \Re z_i < 0 \quad (1 \leq i \leq d). \tag{19}$$

The negative real parts ensure convergence near the boundary, where h need not vanish.

Lemma 4.3. The test transform defines a continuous linear map

$$\mathcal{H} \longrightarrow \Omega^0_{\mathrm{rd}}(A, \mathcal{P}_0), \qquad h \longmapsto H_h.$$

The pole order of H_h is at most one along each hyperplane.

For $U \in \mathcal{A}$, set

$$\beta_{U,h,f} = \langle M_U(z), f\rangle H_h(z)\, dz_1 \wedge \cdots \wedge dz_d.$$

Multiplying (14) by the rapid decay of H_h gives $\beta_{U,h,f} \in \Omega^d_{\mathrm{rd}}(A, \mathcal{P})$.

For iterated residues, each flag starts with the orientation inherited from the preceding flag. If $\mathcal{R}$ denotes such a composition with its sequence of face flags recorded, write $J_{F,\mathcal{R}}$ for the resulting period on its terminal flat.

Lemma 4.4. Fix $U \in \mathcal{A}$ and a finite composition of residue maps followed by a period. Let $B(h, f)$ be the scalar obtained by applying this composition to $\beta_{U,h,f}$. There are continuous seminorms p on $\mathcal{H}$ and q on $\mathcal{F}$ such that

$$|B(h, f)| \le p(h)q(f). \tag{20}$$

Consequently,

$$\langle C(h), f\rangle := B(h, f)$$

defines $C \in \mathcal{S}'_Q(\mathcal{F}'_\beta)$.

Proof. On any fixed contour region Ω with compact real parts and a positive distance from the poles, and for any prescribed L, Lemma 4.3 and (14) give continuous seminorms p, q with

$$|\beta_{U,h,f}(z)| \le p(h)q(f)(1 + \|\Im z\|)^{-L}.$$

By [3, "Residues on the decay complex"], residues can be computed on transverse tori of fixed radii that remain away from the poles. Points on these tori stay a bounded distance from their base points, so taking residues preserves the decay exponent L. After finitely many residues, the same bound holds on each induced contour region, with possibly different continuous seminorms p, q. Parametrize the period contour by $\xi \in \mathbb{R}^r$ and choose $L > r$. The residue bound gives a constant $c > 0$ such that

$$|B(h, f)| \le c\, p(h)q(f) \int_{\mathbb{R}^r} (1 + \|\xi\|)^{-L}\, d\xi < \infty.$$

Absorbing c and the integral into p proves (20). For each h, the linear functional $f \mapsto B(h, f)$ is therefore continuous on $\mathcal{F}$, defining $C(h) \in \mathcal{F}'$. The map $h \mapsto C(h)$ is linear, and for every bounded $K \subset \mathcal{F}$,

$$\sup_{f\in K} |\langle C(h), f\rangle| \le p(h) \sup_{f\in K} q(f) < \infty.$$

This proves continuity into $\mathcal{F}'_\beta$. If h is supported outside Q, then $H_h = 0$, so $B(h, f) = 0$ for every f and hence $C(h) = 0$. Thus $C \in \mathcal{S}'_Q(\mathcal{F}'_\beta)$. □

Write $\beta_{U,h} = M_U H_h\, dz_1 \wedge \cdots \wedge dz_d$, understanding residues and periods through these scalar pairings. In particular $J_{F,\mathcal{R}}(\mathcal{R}[\beta_{U,h}])$ denotes the functional whose value at f is $J_{F,\mathcal{R}}(\mathcal{R}[\beta_{U,h,f}])$.

Proposition 4.5 (Mellin inversion)**.** Let $U \in \mathcal{A}(\mathcal{P}, \mathfrak{c}_0)$. For every $c \in \mathfrak{c}_0$ and $h \in \mathcal{H}$,

$$\langle EU, h\rangle = \frac{1}{(2\pi i)^d} \int_{c+iA} \beta_{U,h} = J_{\mathfrak{c}_0}([\beta_{U,h}]). \tag{21}$$

For every $\omega \in W$,

$$\langle T_\omega EU, h\rangle = J_{\mathfrak{c}_0}([e^{-\omega(z)}\beta_{U,h}]). \tag{22}$$

Proof. Choose $c \in \mathfrak{c}_0$ and put $g_c = A_c h \in \mathcal{S}(W)$ by Lemma B.2. Changing variables $x = e^y$ gives $\widehat{g}_c(\xi) = H_h(c + i\xi)$. Set $\varphi = (2\pi)^{-d}\widehat{g}_c$; Fourier inversion gives $\widehat{\varphi}(-\cdot) = g_c$. Applying (13) to φ and using (18) therefore gives, after pairing with each $f \in \mathcal{F}$,

$$\langle EU, h\rangle = \langle w_c, g_c\rangle = \frac{1}{(2\pi)^d}\int_A M_U(c + i\xi)H_h(c + i\xi)\, d\xi.$$

For $h_\omega = h \circ D_\omega$, changing variables in (19) gives

$$H_{h_\omega}(z) = e^{-\omega(z)}H_h(z).$$

Apply the first identity to h_ω to obtain the second. □

Example 4.6 (Boundary profiles from an interior distribution)**.** The distribution $U = \delta_1'$ on $[0, \infty)$ has $EU = U$ and $M_U(z) = -z$. For $\omega = t \to \infty$, direct computation gives

$$\langle T_t U, h\rangle = -e^{-t}h'(e^{-t}), \qquad e^t T_t U \longrightarrow \delta_0' \quad \text{in } \mathcal{S}_Q'(\mathcal{F}_\beta').$$

We can recover this asymptotic through Mellin inversion as follows. For any $c < 0$,

$$\langle T_t U, h\rangle = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(-z)e^{-tz}H_h(z)\, dz.$$

To identify the poles of H_h, split its defining integral at $x = 1$ and use $h(x) = h(0) + xh'(0) + O(x^2)$ near zero. Integrating the first two Taylor terms gives the continuation

$$H_h(z) = -\frac{h(0)}{z} + \frac{h'(0)}{1 - z} + G(z),$$

where G is holomorphic for $\Re z < 2$. So

$$\operatorname{Res}_{z=1} H_h(z) = -h'(0), \qquad \operatorname{Res}_{z=1}\big((-z)e^{-tz}H_h(z)\big) = e^{-t}h'(0).$$

The factor $-z$ cancels the pole at $z = 0$. So with both vertical contours oriented upward, shifting to $1 < \sigma < 2$ subtracts the residue at $z = 1$ and gives

$$\langle T_t U, h\rangle = -e^{-t}h'(0) + \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}(-z)e^{-tz}H_h(z)\, dz.$$

For every bounded test set $\mathcal{B}$, the rapid vertical decay of H_h from Lemma 4.3 bounds the remaining integral by $O_{\mathcal{B}}(e^{-\sigma t})$. Consequently,

$$e^t T_t U \longrightarrow \delta_0' \quad \text{in } \mathcal{S}_Q'(\mathcal{F}_\beta'),$$

with the limit arising from the pole at $z = 1$.

5 Residue coefficients and contour bounds

5.1 Contour bounds and face exponents

On a vertical contour based at c,

$$|e^{-\omega(c+i\xi)}| = e^{-\omega(c)}.$$

Larger values of $\omega(c)$ improve the exponential factor in the contour bound. By [3, "Rapid decay cohomology and periods"], the period is unchanged as c moves within its face F, so the upper limit of the available exponential decay rates is

$$\lambda_F(\omega) = \sup_{x \in \overline{F}} \omega(x)$$

for a real affine weight ω. This is the linear program (5) on $\mathrm{aff}(F)$, applying both to chambers and to faces carrying residue periods. When ω is nonconstant on F and the LP value is finite, the maximizing face marks where further improvement requires crossing poles, which can be performed using the jump formula (6). A value of $+\infty$ means that any finite target for the contour exponent can be exceeded.

Fix a nonzero linear weight θ and a threshold T. Theorem 3.1 produces residue faces F on which θ is constant with value at most T, and remainder chambers in which we choose contour bases c_ℓ with $\theta(c_\ell) > T$. Thus there is a strict gap between the residue and remainder exponents.

5.2 Residue coefficients as functions of the weight

Fix a finite jump expansion, independently of the input $U \in \mathcal{A}(\mathcal{P}, \mathfrak{c}_0)$, keeping all its flags and integer coefficients fixed. Combining the residue terms ending at a face F defines an operator $C_F : \mathcal{A}(\mathcal{P}, \mathfrak{c}_0) \to \mathcal{S}'_Q(\mathcal{F}'_\beta)$. It and its dilates satisfy, for every $\omega \in W$,

$$\begin{aligned} \langle C_F U, h \rangle &= \sum_p a_p J_{F,\mathcal{R}_p}(\mathcal{R}_p[\beta_{U,h}]), \\ \langle T_\omega C_F U, h \rangle &= \sum_p a_p J_{F,\mathcal{R}_p}(\mathcal{R}_p[e^{-\omega}\beta_{U,h}]), \end{aligned} \tag{23}$$

where each branch gives a composition $\mathcal{R}_p : H^d_{\mathrm{rd}}(A, \mathcal{P}) \to H^{\dim F}_{\mathrm{rd}}(\mathrm{aff}(F), \mathcal{P}_{\mathrm{aff}(F)})$, with its sequence of face flags recorded. The periods use the orientation induced along each branch, and their values are summed in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$. The second identity follows from $H_{h \circ D_\omega} = e^{-\omega} H_h$.

Write $\mathrm{Pol}(B, \mathcal{S}'_Q(\mathcal{F}'_\beta))$ for the space of $\mathcal{S}'_Q(\mathcal{F}'_\beta)$-valued polynomials of arbitrary finite degree on an affine space B. In affine coordinates this means a finite sum $\sum_\gamma x^\gamma V_\gamma$ with fixed distributions $V_\gamma \in \mathcal{S}'_Q(\mathcal{F}'_\beta)$. Convergence coefficientwise means convergence of each V_γ in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$ after using the same coordinates. For $X \in W_F$, the weight is constant on F with value $\ell_F(X)$. Define

$$P_F : \mathcal{A}(\mathcal{P}, \mathfrak{c}_0) \longrightarrow \mathrm{Pol}(W_F, \mathcal{S}'_Q(\mathcal{F}'_\beta))$$

by

$$(P_F U)(X) = e^{\ell_F(X)} T_X C_F U, \qquad X \in W_F. \tag{24}$$

For a nonempty flat L of $\mathcal{P}$, put

$$m_U(L) := \sum_{\substack{H \in \mathcal{P} \\ L \subseteq H}} \big(\mathrm{poleord}_H(M_U) + \mathbf{1}_{\mathcal{P}_0}(H)\big), \tag{25}$$

where $\mathrm{poleord}_H(M_U)$ is the least common upper bound for the pole orders of $\langle M_U, f \rangle$, $f \in \mathcal{F}$, along $H_{\mathbb{C}}$, and is zero when all these pairings are holomorphic there. The sum is finite by local finiteness. Its summands bound the pole orders of $M_U H_h$ uniformly in h and f by Lemma 4.3.

Lemma 5.1. The maps C_F and P_F are linear. For every $U \in \mathcal{A}(\mathcal{P}, \mathfrak{c}_0)$,

$$P_F U \in \mathrm{Pol}(W_F, \mathcal{S}'_Q(\mathcal{F}'_\beta)).$$

Its degree satisfies

$$\deg(P_F U) \le m_U(\mathrm{aff}(F)) - \operatorname{codim} F. \tag{26}$$

If the right-hand side is negative, then $P_F U = 0$. For fixed U, the polynomials

$$W_F \longrightarrow \mathcal{S}'_Q(\mathcal{F}'_\beta), \qquad X \longmapsto T_b((P_F U)(X))$$

satisfy the same degree bound independently of b, and their coefficients depend continuously on b in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$.

For fixed U, the space

$$E_{F,U} = \mathrm{span}\{(P_F U)(X) : X \in W_F\} \subseteq \mathcal{S}'_Q(\mathcal{F}'_\beta)$$

is finite dimensional. Fix a norm on $E_{F,U}$. For every $c \in F$, there is an integer $D \ge 0$ such that, for every bounded $\mathcal{B} \subset \mathcal{H}$,

$$\|T_{X+b} C_F U\|_{\mathcal{B}} \le C_{U,\mathcal{B},c}(1 + \|b\|)^D e^{-(X+b)(c)} \|(P_F U)(X)\|_{E_{F,U}}, \tag{27}$$

for all $X \in W_F$ and $b \in W$.

Proof. The residue and period maps are linear, so C_F and P_F are linear in U.

Fix U and put $L = \mathrm{aff}(F)$. By the composition property of residues in [3, "The flag residue"], each branch operator $\mathcal{R}_p$ is the residue along its combined flag of affine flats. We compute it using a compatible choice of projections.

By Cauchy's integral formula, a codimension-one residue at a pole of order a is

$$\mathrm{Res}_{t=0} \frac{\eta(y,t) \wedge dt}{t^a} = \frac{1}{(a-1)!} \partial_t^{a-1} \eta(y, 0).$$

Each step therefore uses finitely many normal derivatives. A flag residue, being a finite composition of these steps, depends only on a finite-order normal jet of the numerator after clearing its poles.

Put $r = \operatorname{codim} L$ and $m = m_U(L)$. Near a regular point $y \in L_{\mathbb{C}}$, write the normal coefficient as $g(y,n)/Q(n)$, where g is holomorphic and Q is the product of the polar factors containing L, with the multiplicities in (25). Thus Q is homogeneous of degree m; the remaining polar factors are nonzero near $(y, 0)$ and are absorbed into g. Finite-jet dependence allows us to replace g by a finite Taylor polynomial in n. Under $n \mapsto \rho n$, a homogeneous numerator term of degree k contributes a factor ρ^{k+r-m} to the residue integral. Dilating the flag torus preserves its homology class, so the residue is unchanged. Hence only degree $m - r$ contributes, and the residue vanishes if $m < r$.

For the weighted form the numerator is $e^{-\omega(y)} e^{-d\omega(n)} g(y,n)$. When $m \ge r$, only powers of $d\omega(n)$ through degree $m - r$ can contribute. Thus, after factoring out $e^{-\omega|_L}$, the residue is polynomial in ω of degree at most $m - r$. This local identity extends meromorphically along $L_{\mathbb{C}}$ and gives

$$\mathcal{R}_p[e^{-\omega} \beta_{U,h}] = \left[e^{-\omega|_L} \sum_\alpha q_{p,\alpha}(\omega) \gamma_{p,\alpha}(h) \right]. \tag{28}$$

Here $\deg q_{p,\alpha} \le m_U(L) - \operatorname{codim} L$, independently of h and f; if this bound is negative, the residue is zero. For a branch with no residues, the polynomial is constant and the bound is zero. The

meromorphic top forms $\gamma_{p,\alpha}(h)$ on $L_{\mathbb{C}}$ are independent of ω and linear in h. Lemma 4.3 and the residue bounds of [3, "Residues on the decay complex"] give their rapid vertical decay, with bounds controlled by products of continuous seminorms of h and f, as in Lemma 4.4.

For $X \in W_F$, the factor $e^{-X|_L} = e^{-\ell_F(X)}$ cancels the normalization in P_F. Hence

$$\langle (P_F U)(X), h\rangle = \sum_{p,\alpha} a_p q_{p,\alpha}(X) J_{F,\mathcal{R}_p}([\gamma_{p,\alpha}(h)]).$$

By [3, "Rapid decay cohomology and periods"] and these bounds, each period on the right is a continuous linear map $\mathcal{H} \to \mathcal{F}'_\beta$, so this is a polynomial in X with coefficients in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$. Restriction to W_F, period integration, and finite summation preserve the degree bound, proving (26). These distributions are supported in Q, since $H_h = 0$ for tests supported outside Q, and therefore belong to $\mathcal{S}'_Q(\mathcal{F}'_\beta)$. Applying T_b to these coefficients preserves the degree bound. Their continuous dependence on b follows from continuity of dilations.

To prove the contour bound, fix $c \in F$. On $c + iV_L$ the exponential in (28) has modulus $e^{-\omega(c)}$. Taking absolute values in the contour integral over $c + iV_L$ and using the decay bounds above gives, for some D independent of $\mathcal{B}$,

$$\|T_\omega C_F U\|_{\mathcal{B}} \le C_{U,\mathcal{B},c}(1 + \|\omega\|)^D e^{-\omega(c)}.$$

More precisely, for every bounded $\mathcal{K} \subset \mathcal{F}$, the same integrable majorants give an order k and a constant C with

$$\sup_{f\in\mathcal{K}} |\langle (T_\omega C_F U)(h), f\rangle| \le C p_k(h)(1 + \|\omega\|)^D e^{-\omega(c)}.$$

Here p_k is the finite Schwartz seminorm in (31). The order k may depend on $\mathcal{K}$, whereas D is bounded by the degrees of the finitely many weight polynomials. We use this estimate to bound T_b on $E_{F,U}$. If $E_{F,U} = 0$, there is nothing to prove. Otherwise choose a basis $e_j = (P_F U)(X_j)$ from its spanning family. Since $T_b e_j = e^{\ell_F(X_j)} T_{b+X_j} C_F U$ and $X_j(c) = \ell_F(X_j)$, the estimate at $\omega = b + X_j$ gives

$$\|T_b e_j\|_{\mathcal{B}} \le C_{U,\mathcal{B},c}(1 + \|b + X_j\|)^D e^{-b(c)}.$$

The finitely many X_j are fixed, so expansion in this basis gives

$$\|T_b v\|_{\mathcal{B}} \le C_{U,\mathcal{B},c}(1 + \|b\|)^D e^{-b(c)} \|v\|_{E_{F,U}}, \qquad v \in E_{F,U}.$$

Apply this bound to $v = (P_F U)(X)$ and multiply by $e^{-\ell_F(X)}$. The definition of P_F and the group law give $e^{-\ell_F(X)} T_b v = T_{X+b} C_F U$; since $X(c) = \ell_F(X)$, this proves (27). □

5.3 A double pole in two variables

A double pole makes the polynomial factor in Lemma 5.1 explicit. Let

$$U = -\log x \, \mathbf{1}_{\{0<x<1,\ 0<xy<1\}} \, dx\, dy.$$

This is a positive tempered measure with $EU = U$. For $\Re w > -1$ and $\Re(z - w) > 0$, its Mellin transform is

$$\begin{aligned} M_U(z,w) &= \int_0^1 (-\log x) x^z \left(\int_0^{1/x} y^w \, dy \right) dx \\ &= \frac{1}{w+1} \int_0^1 (-\log x) x^{z-w-1} \, dx \\ &= \frac{1}{(w+1)(z-w)^2}. \end{aligned} \tag{29}$$

For $\omega = (-t, -s)$, choose $-1 < b < a < 0$. Mellin inversion gives

$$\langle T_{(-t,-s)}U, h\rangle = \frac{1}{(2\pi i)^2}\int_{\Re w=b}\int_{\Re z=a}\frac{e^{tz+sw}H_h(z,w)}{(w+1)(z-w)^2}\,dz\,dw.$$

Move the z-contour left across $z = w$. The residue at this double pole is

$$\begin{aligned}\operatorname{Res}_{z=w}\frac{e^{tz+sw}H_h(z,w)}{(w+1)(z-w)^2} &= \frac{1}{w+1}\partial_z\big(e^{tz+sw}H_h(z,w)\big)\Big|_{z=w}\\ &= \frac{e^{(t+s)w}}{w+1}\big[tH_h(w,w) + \partial_z H_h(w,w)\big].\end{aligned}$$

The differentiation produces the factor t. Now move the contour of this one-dimensional residue integral left across $w = -1$. Its residue is

$$e^{-t-s}\big[tH_h(-1,-1) + \partial_z H_h(-1,-1)\big].$$

By the definition of the test kernel,

$$H_h(-1,-1) = \int_Q h(x,y)\,dx\,dy, \qquad \partial_z H_h(-1,-1) = -\int_Q \log x\, h(x,y)\,dx\,dy.$$

Thus the iterated residue is $e^{-t-s}\int_Q (t - \log x)h(x,y)\,dx\,dy$. Both contour shifts are to the left, so the residues enter with positive signs. Collecting the remaining contour integrals into $R_{t,s}$ gives

$$T_{(-t,-s)}U = e^{-t-s}(t - \log x)\,dx\,dy + R_{t,s}. \tag{30}$$

Division by te^{-t-s} gives the limit $dx\,dy$ as $t \to \infty$ and $t + s \to \infty$. This can also be checked directly: the density after dilation is $e^{-t-s}(t - \log x)$ on $x < e^t$, $xy < e^{t+s}$.

6 Relative remainder estimates

Theorem 3.1 and Lemma 5.1 give absolute contour bounds. We compare the remainders with the original distribution using finite Schwartz orders. For bounded $\mathcal{K} \subset \mathcal{F}$, put

$$p_m(h) = \max_{|\alpha|\le m}\sup_{x\in Q}(1 + \|x\|)^m|\partial^\alpha h(x)|, \qquad \|V\|_{-m,\mathcal{K}} = \sup_{\substack{p_m(h)\le 1\\ f\in\mathcal{K}}}|\langle V(h), f\rangle|. \tag{31}$$

For each fixed $\mathcal{K}$, continuity of V makes this finite at all sufficiently large orders. More generally, if (V_i) is bounded in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$, then Banach–Steinhaus on $\mathcal{H}$, applied to the scalar functionals $h \mapsto \langle V_i(h), f\rangle$ for i and $f \in \mathcal{K}$, gives an m with $\sup_i \|V_i\|_{-m,\mathcal{K}} < \infty$. For $\mathcal{B} = A_0 \times \mathcal{K}$ bounded,

$$\|V\|_{\mathcal{B}} \le \big(\sup_{h\in A_0} p_m(h)\big)\|V\|_{-m,\mathcal{K}}.$$

Thus an estimate in one order for each bounded $\mathcal{K}$ proves convergence in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$; the order may depend on $\mathcal{K}$.

We will also use finite test sets to detect finite-dimensional spaces. For every finite-dimensional $\mathscr{V} \subset \mathcal{S}'_Q(\mathcal{F}'_\beta)$, there are pairs (h_j, f_j) such that

$$\|v\|_{\mathscr{V}} := \max_j |\langle v(h_j), f_j\rangle| \tag{32}$$

is a norm on $\mathscr{V}$. Indeed, all such evaluations separate operators, and finitely many suffice in finite dimension. If $\{f_j\} \subseteq \mathcal{K}$, then $\|v\|_{\mathscr{V}} \le C_m \|v\|_{-m,\mathcal{K}}$. For sufficiently large m the reverse bound holds on $\mathscr{V}$ by continuity and finite dimension. We keep these finite detecting sets fixed before choosing any Schwartz orders.

Dilation preserves finiteness at the same order. Indeed, the chain rule gives constants C_m, a_m such that

$$p_m(h \circ D_\omega) \le C_m e^{a_m \|\omega\|} p_m(h). \tag{33}$$

and hence $\|T_\omega V\|_{-m,\mathcal{K}} \le C_m e^{a_m\|\omega\|} \|V\|_{-m,\mathcal{K}}$. Applying the same bound to the inverse dilation gives the converse.

6.1 Quotients and fixed complements

Only finitely many spaces W_F occur, since each is spanned by a subset of the finitely many polar normal directions. Let $\mathcal{S}_0$ be the finite collection of all intersections of these spaces. For a subspace $L \subseteq W$, let $q_L : W \to W/L$ be the quotient map, with the quotient norm, so $s_F(\omega) = \|q_{W_F}\omega\|$. For each $S \in \mathcal{S}_0$, fix any complement B_S and let

$$W = S \oplus B_S, \qquad p_S : W \to S, \qquad b_S = I - p_S : W \to B_S$$

be the associated projections. Write

$$\omega = Y + b, \qquad Y = p_S\omega, \quad b = b_S\omega.$$

The map $q_S|_{B_S} : B_S \to W/S$ is an isomorphism. Thus quotient convergence is equivalent to convergence of the chosen representatives $b_S\omega$, and their norms are comparable. All complements are fixed before choosing an input or a net; constants may depend on them. Define

$$\mathcal{I}_S = \{F : S \subset W_F\}, \qquad \mu_S(\omega) = \min_{F \notin \mathcal{I}_S} s_F(\omega). \tag{34}$$

The faces in $\mathcal{I}_S$ are those on which every weight in S is constant.

Lemma 6.1. Every net $(\omega_i)_{i\in I}$ in W with $\|\omega_i\| \to \infty$ has a subnet and an $S \in \mathcal{S}_0 \setminus \{0\}$ such that

$$b_S\omega_i \longrightarrow b \in B_S, \qquad \mu_S(\omega_i) \longrightarrow \infty.$$

Proof. Using the one-point compactifications of the finitely many nonzero quotient spaces W/W_F, pass to a subnet on which each distinct quotient class $q_{W_F}\omega_i$ converges in its quotient space or diverges in norm. The zero quotient is constant. Let $L_1, \dots, L_r$ be the distinct spaces W_F with convergent quotient classes, and put

$$S = \bigcap_{j=1}^{r} L_j \in \mathcal{S}_0,$$

with an empty intersection interpreted as W. The linear map

$$W/S \longrightarrow \bigoplus_{j=1}^{r} W/L_j, \qquad q_S\omega \longmapsto (q_{L_j}\omega)_j$$

is injective, with closed image and continuous inverse on its image. Since every component converges, $q_S\omega_i$ converges, and hence $b_S\omega_i \to b \in B_S$. If $S = 0$, this would make ω_i converge, a contradiction.

For $F \notin \mathcal{I}_S$, the space W_F is not among the L_j, since $S \subseteq L_j$ for every j. Hence $s_F(\omega_i) = \|q_{W_F}\omega_i\| \to \infty$. There are only finitely many distinct quotient maps, so $\mu_S(\omega_i) \to \infty$. □

6.2 Residue periods and coefficient extraction

Definition 6.2. Fix the polar arrangement $\mathcal{P}$ and the input space $\mathcal{A} = \mathcal{A}(\mathcal{P}, \mathfrak{c}_0)$. The generating data are:

(i) an input $U \in \mathcal{A}$;

(ii) a chain of affine flats $L_0 = A \supsetneq L_1 \supsetneq \cdots \supsetneq L_s$, where $0 \le s \le d$, and, for each $i = 1, \dots, s$, a face flag FL_i of $\mathcal{P}_{L_{i-1}}$ whose base has affine hull L_i;

(iii) a chamber F of the induced arrangement $\mathcal{P}_{L_s}$.

For these data, put

$$\mathcal{R} = \mathrm{Res}_{FL_s} \circ \cdots \circ \mathrm{Res}_{FL_1} : H^d_{\mathrm{rd}}(A, \mathcal{P}) \longrightarrow H^{\dim L_s}_{\mathrm{rd}}(L_s, \mathcal{P}_{L_s}),$$

with the sequence $(FL_1, \dots, FL_s)$ recorded as part of $\mathcal{R}$, and with $\mathcal{R}$ the identity when $s = 0$. The terminal flat carries the orientation induced successively by these flags. Define the distribution $V_{U,F,\mathcal{R}} \in \mathcal{S}'_Q(\mathcal{F}'_\beta)$ by

$$\langle V_{U,F,\mathcal{R}}, h \rangle = J_{F,\mathcal{R}}(\mathcal{R}[\beta_{U,h}]) \qquad (h \in \mathcal{H}).$$

The test-function bounds in Lemma 5.1 make this pairing continuous. If h is supported outside Q, then $H_h = 0$ and the pairing vanishes, so the resulting distribution is supported in Q. Define

$$\mathscr{P} := \mathrm{span}_{\mathbb{C}}\{V_{U,F,\mathcal{R}} : \text{the data satisfy (i)–(iii)}\} \subseteq \mathcal{S}'_Q(\mathcal{F}'_\beta). \tag{35}$$

A residue period map on F is a map of the form

$$\mathcal{A} \longrightarrow \mathscr{P}, \qquad U \longmapsto \sum_{p=1}^{N} a_p V_{U,F,\mathcal{R}_p}, \qquad a_p \in \mathbb{C}.$$

Taking $s = 0$ and $F = \mathfrak{c}_0$ gives $V_{U,\mathfrak{c}_0,I} = EU$ by Proposition 4.5, so $E\mathcal{A} \subseteq \mathscr{P}$. For $\eta \in W$, $H_{h\circ D_\eta} = e^{-\eta} H_h$ and $M_{T_\eta U} = e^{-\eta} M_U$, so $\beta_{U,h\circ D_\eta} = \beta_{T_\eta U,h}$. Thus

$$\begin{aligned} \langle T_\eta V_{U,F,\mathcal{R}}, h \rangle &= J_{F,\mathcal{R}}(\mathcal{R}[\beta_{U,h\circ D_\eta}]) \\ &= J_{F,\mathcal{R}}(\mathcal{R}[\beta_{T_\eta U,h}]) = \langle V_{T_\eta U,F,\mathcal{R}}, h \rangle. \end{aligned} \tag{36}$$

Hence $T_\eta V_{U,F,\mathcal{R}} = V_{T_\eta U,F,\mathcal{R}}$, and fixed dilations restrict to maps $T_\eta : \mathscr{P} \to \mathscr{P}$. Appending another flag residue to $\mathcal{R}$ gives generating data of the same kind on the smaller flat.

For a subspace $K \subseteq W$, the K-dilation span of $A \subseteq \mathcal{S}'_Q(\mathcal{F}'_\beta)$ is

$$\mathrm{span}_{\mathbb{C}}\{T_Y v : Y \in K,\ v \in A\}.$$

We say that $v \in \mathcal{S}'_Q(\mathcal{F}'_\beta)$ has character $\beta \in K^*$ on K if the map

$$K \longrightarrow \mathcal{S}'_Q(\mathcal{F}'_\beta), \qquad Y \longmapsto e^{\beta(Y)} T_Y v$$

is polynomial. For a residue period on a face F with $K \subseteq W_F$, Lemma 5.1 gives this property with $\beta(Y) = Y|_F$. Its K-dilation span is finite dimensional: the values of this polynomial span the same space as its finitely many coefficients, since each coefficient of a vector-valued polynomial is a finite linear combination of its values.

Fix $K \subseteq W$. For every $\beta \in K^*$, let $E^K_\beta \subseteq \mathcal{S}'_Q(\mathcal{F}'_\beta)$ be the space of distributions with character β on K, and put

$$E^K = \sum_{\beta \in K^*} E^K_\beta.$$

Lemma 6.3. The sum is direct:

$$E^K = \bigoplus_{\beta \in K^*} E^K_\beta.$$

For each $\alpha \in K^*$, the projection

$$\mathsf{P}_\alpha : E^K \longrightarrow E^K_\alpha \tag{37}$$

commutes with every dilation. On the W-dilation span $\mathscr{M}$ of any finite family of distributions with characters on K, it satisfies, for every bounded $\mathcal{K} \subset \mathcal{F}$ and every m,

$$\|\mathsf{P}_\alpha v\|_{-m,\mathcal{K}} \le C_{\alpha,m,\mathscr{M}} \|v\|_{-m,\mathcal{K}} \qquad (v \in \mathscr{M}).$$

Proof. Consider a finite relation $\sum_{\beta\in\Lambda} v_\beta = 0$ with $v_\beta \in E^K_\beta$. Choose $X \in K$ such that the values $\beta(X)$ are distinct. After applying T_{tX} and pairing with $h \in \mathcal{H}$ and $f \in \mathcal{F}$, we obtain a zero sum of polynomials times distinct real exponentials. Multiply by the exponential corresponding to the smallest $\beta(X)$. Its polynomial tends to zero as $t \to \infty$, so it is zero. Remove it and repeat. Thus every $v_\beta = 0$, proving the direct sum. Since dilations commute, each E^K_β is invariant under every T_Z, $Z \in W$. Its coordinate projection therefore commutes with every dilation.

For the estimate, choose generators $v_j \in E^K_{\beta_j}$ for $\mathscr{M}$ under W-dilation, put $\Lambda = \{\beta_j\}$, and choose $X \in K$ for which the $\beta(X)$ are distinct. Let D_β bound the polynomial degrees of $Y \mapsto e^{\beta(Y)} T_Y v_j$ for the generators with character β. Commutation of dilations preserves these degree bounds throughout $\mathscr{M}$. For $v \in \mathscr{M}$, write $v = \sum_{\beta\in\Lambda} \mathsf{P}_\beta v$. The polynomial $e^{\beta(X)t} T_{tX} \mathsf{P}_\beta v$ has constant term $\mathsf{P}_\beta v$, as one sees by setting $t = 0$. Consequently, for every $h \in \mathcal{H}$ and $f \in \mathcal{F}$,

$$\langle (T_{tX} v)(h), f\rangle = \sum_{\beta\in\Lambda} e^{-\beta(X)t} \left(\langle (\mathsf{P}_\beta v)(h), f\rangle + \sum_{k=1}^{D_\beta} a_{\beta,k}(v,h,f) t^k \right),$$

where the $a_{\beta,k}(v,h,f)$ are scalar coefficients. Thus $\langle (\mathsf{P}_\alpha v)(h), f\rangle$ is the coefficient of $t^0 e^{-\alpha(X)t}$ in a function belonging to the fixed finite-dimensional space

$$\mathscr{E} = \operatorname{span}_{\mathbb{C}} \{ t^k e^{-\beta(X)t} : \beta \in \Lambda,\ 0 \le k \le D_\beta \}.$$

Evaluations at $t \ge 0$ separate functions in $\mathscr{E}$, so finitely many points $t_1, \dots, t_N$ give a norm $f \mapsto \max_i |f(t_i)|$ on $\mathscr{E}$. Every coefficient is a linear functional on $\mathscr{E}$ and is therefore bounded in this norm. Applying this to the displayed coefficient gives, for $\alpha \in \Lambda$,

$$|\langle (\mathsf{P}_\alpha v)(h), f\rangle| \le C \max_i |\langle (T_{t_i X} v)(h), f\rangle|.$$

Here C and the points t_i are independent of v, h, and f. Taking the supremum over $p_m(h) \le 1$ and $f \in \mathcal{K}$, then bounding each of the finitely many fixed dilations by (33), yields

$$\|\mathsf{P}_\alpha v\|_{-m,\mathcal{K}} \le C \max_i \|T_{t_i X} v\|_{-m,\mathcal{K}} \le C_{\alpha,m,\mathscr{M}} \|v\|_{-m,\mathcal{K}}.$$

For $\alpha \notin \Lambda$, the projection is zero on $\mathscr{M}$. □

Lemma 6.4. Let $K' \subseteq W$, $0 \ne \theta \in K'$, and let $\Lambda \subset (K')^*$ be finite. Suppose $v_\beta \in E^{K'}_\beta$ for $\beta \in \Lambda$ and $r \in \mathcal{S}'_Q(\mathcal{F}'_\beta)$ satisfy

$$0 = \sum_{\beta\in\Lambda} v_\beta + r, \qquad T_{t\theta} r = O_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(e^{-At}).$$

Then $v_\alpha = 0$ for every $\alpha \in \Lambda$ with $\alpha(\theta) < A$.

Proof. Since $r = -\sum_\beta v_\beta \in E^{K'}$, all terms lie in the W-dilation span of the finitely many v_β. Apply P_α to the dilated identity. The finite-dilation interpolation formula in Lemma 6.3 preserves bounds for each seminorm in (10) after enlarging its bounded Schwartz test set, and therefore preserves the remainder bound and annihilates the other groups, giving

$$e^{-t\alpha(\theta)}Q_\alpha(t) = O_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(e^{-At}), \qquad Q_\alpha(0) = v_\alpha,$$

where Q_α is a distribution-valued polynomial. Each scalar test pairing of $Q_\alpha(t)$ is a scalar polynomial bounded by a constant times $e^{-(A-\alpha(\theta))t}$, so it is zero. Thus $Q_\alpha = 0$ and $v_\alpha = 0$. □

6.3 Canonical coefficient maps

The normal arrangement $\mathcal{Q}$ in W consists of the hyperplanes spanned by $d-1$ linearly independent normal directions of the polar arrangement $\mathcal{P}$.

Let $S \subseteq W$ and let C be a nonzero full-dimensional cone of the restriction of $\mathcal{Q}$ to S. On the space $\mathscr{P}$ of Definition 6.2, define for $\alpha \in S^*$

$$\mathcal{C}_\alpha^{S,C} : \mathscr{P} \longrightarrow \mathscr{P} \cap E_\alpha^S$$

as follows. Write $v = \sum_{j=1}^n c_j V_j$, where the V_j are residue periods from Definition 6.2. Choose $\theta \in C$ and apply Theorem 3.1 and Lemma 5.1 to each V_j with a common threshold $T > \alpha(\theta)$:

$$V_j = \sum_\ell W_{j\ell} + R_j, \qquad T_{t\theta}R_j = O_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(e^{-Tt}).$$

Here $W_{j\ell} \in \mathscr{P}$ is a residue term on a face $F_{j\ell}$ on which θ is constant. Lemma A.1 implies that every $Y \in S$ is constant on this face, so Lemma 5.1 gives $W_{j\ell}$ the character $\beta_{j\ell} \in S^*$ given by $\beta_{j\ell}(Y) = Y|_{F_{j\ell}}$. If θ is constant on the source face of V_j, take $W_{j1} = V_j$ and $R_j = 0$. Set

$$\mathcal{C}_\alpha^{S,C} v := \sum_{j=1}^n c_j \sum_{\ell:\, \beta_{j\ell}=\alpha} W_{j\ell}.$$

Lemma 6.5. The maps $\mathcal{C}_\alpha^{S,C} : \mathscr{P} \to \mathscr{P} \cap E_\alpha^S$ are linear projections onto $\mathscr{P} \cap E_\alpha^S$, independent of the decomposition $v = \sum_{j=1}^n c_j V_j$, the objective $\theta \in C$, and the threshold. They commute with every dilation and satisfy

$$\mathcal{C}_\alpha^{S,C} v = \begin{cases} v, & \beta = \alpha, \\ 0, & \beta \neq \alpha, \end{cases} \qquad (v \in \mathscr{P} \cap E_\beta^S).$$

For every subspace $K \subseteq W$ and $\chi \in K^*$,

$$\mathcal{C}_\alpha^{S,C}(\mathscr{P} \cap E_\chi^K) \subseteq \mathscr{P} \cap E_\chi^K \cap E_\alpha^S.$$

If $v \in \mathscr{P}$ satisfies $T_{t\theta}v = O_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(e^{-At})$ for some $\theta \in C$ and $A > \alpha(\theta)$, then $\mathcal{C}_\alpha^{S,C} v = 0$. For finite-dimensional $\mathscr{V} \subseteq \mathscr{P}$, the S-dilation span of $\mathcal{C}_\alpha^{S,C}(\mathscr{V})$ is finite dimensional.

Additionally, let

$$v = \sum_j w_j + \sum_\ell r_\ell, \qquad w_j \in \mathscr{P} \cap E_{\chi_j}^K, \qquad K \subseteq S,$$

where each r_ℓ is a residue period with contour base c_ℓ. If there exists $\eta \in \overline{C}$ such that $\eta(c_\ell) > \alpha(\eta)$ for every ℓ, then

$$\mathcal{C}_\alpha^{S,C} v = \sum_{j:\, \chi_j = \alpha|_K} \mathcal{C}_\alpha^{S,C} w_j. \tag{38}$$

Proof. Every term selected in the definition belongs to $\mathscr{P} \cap E^S_\alpha$, so their sum does too. Fix $\theta \in C$. Two decompositions of v, expanded with thresholds $T_1, T_2 > \alpha(\theta)$, give a difference

$$0 = \sum_\beta u_\beta + r, \qquad u_\beta \in \mathscr{P} \cap E^S_\beta, \qquad T_{t\theta} r = O_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(e^{-\min(T_1,T_2)t}).$$

Here u_α is the difference between the two proposed values of $\mathcal{C}^{S,C}_\alpha v$. Lemma 6.4, with $K' = S$, gives $u_\alpha = 0$. This proves independence of the decomposition and threshold. To expand a linear combination of inputs, use a common threshold and combine their expansions; the defining sum then gives linearity.

For one fixed expansion, choose its remainder bases c_ℓ. The strict inequalities $\theta(c_\ell) > \alpha(\theta)$ persist for nearby $\vartheta \in C$. Lemma 5.1 still bounds the remainder by a rate faster than $e^{-t\alpha(\vartheta)}$, so Lemma 6.4, applied to an expansion at ϑ, gives the same coefficient. Thus the coefficient is locally constant in θ, and hence constant on the connected cone C.

For $X \in W$, dilation maps $\mathscr{P} \cap E^S_\beta$ into itself by (36) and preserves the remainder estimate by (33). Lemma 6.4, applied to the dilation of an expansion of v and an expansion of $T_X v$, therefore gives

$$\mathcal{C}^{S,C}_\alpha T_X v = T_X \mathcal{C}^{S,C}_\alpha v.$$

If $v \in \mathscr{P} \cap E^S_\beta$, compare its defining expansion with the single term v and zero remainder. Lemma 6.4 gives $\mathcal{C}^{S,C}_\alpha v = v$ when $\beta = \alpha$ and $\mathcal{C}^{S,C}_\alpha v = 0$ otherwise.

If $v \in \mathscr{P} \cap E^K_\chi$, the map $Y \mapsto e^{\chi(Y)} T_Y v$ is polynomial on K, with coefficients in $\mathscr{P}$. Applying $\mathcal{C}^{S,C}_\alpha$ to its coefficients gives the polynomial

$$Y \longmapsto \mathcal{C}^{S,C}_\alpha\big(e^{\chi(Y)} T_Y v\big) = e^{\chi(Y)} T_Y \mathcal{C}^{S,C}_\alpha v, \qquad Y \in K.$$

Thus $\mathcal{C}^{S,C}_\alpha v \in E^K_\chi$; together with the codomain already proved, this gives the asserted inclusion.

To prove the assertion about decay, expand v at θ with threshold $B > A$, writing $v = \sum_\beta u_\beta + r$. Then $0 = \sum_\beta u_\beta + (r - v)$ has remainder $O_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(e^{-At})$. Lemma 6.4 gives $u_\alpha = \mathcal{C}^{S,C}_\alpha v = 0$.

For (38), choose $\theta \in C$ close enough to η that every strict remainder inequality persists. The contour bounds of Lemma 5.1 absorb their polynomial factors into these gaps, so the decay criterion just proved gives $\mathcal{C}^{S,C}_\alpha r_\ell = 0$. Moreover, $\mathcal{C}^{S,C}_\alpha w_j$ has characters χ_j and $\alpha|_K$ on K, so it is zero unless they agree, by Lemma 6.3. Linearity gives the formula.

Finally, the S-dilation span of each $\mathcal{C}^{S,C}_\alpha v$ is finite dimensional by its polynomial character law. Summing these spans over a basis of $\mathscr{V}$ proves the last finiteness assertion. □

6.4 Coefficients on complementary subspaces

Fix S and a full-dimensional cone C of $\mathcal{Q}|_S$. Choose defining forms $\mathcal{N}_S$ for the hyperplanes of $\mathcal{Q}|_S$, oriented so that $\ell > 0$ on C.

For a flat $K \subseteq S$ of $\mathcal{Q}|_S$, define the quotient cone

$$D_K = \{\bar{y} \in S/K : \bar{\ell}(\bar{y}) > 0 \text{ for every } \ell \in \mathcal{N}_S \text{ with } \ell|_K = 0\}, \qquad \bar{\ell}(y + K) = \ell(y).$$

The image of C supplies points satisfying all these strict inequalities, so D_K is full dimensional. Their common kernel in S is K, so $\overline{D}_K$ is pointed. For a complement $S = K \oplus B$, let

$$C_{K,B} = \{y \in B : y + K \in D_K\}.$$

Call B adapted to (K, C) if

$$\ell(y) > 0 \qquad \big(y \in \overline{C_{K,B}} \setminus \{0\},\ \ell \in \mathcal{N}_S,\ \ell|_K \neq 0\big). \tag{39}$$

Lemma 6.6. If there exists $\xi \in K$ such that $\ell(\xi) > 0$ whenever $\ell|_K \neq 0$, then an adapted complement exists. Moreover, suppose B is adapted to (K, C) and Δ is a nonzero relatively open face of $\overline{C_{K,B}}$. Put

$$H = \operatorname{span} \Delta, \qquad K' = K \oplus H.$$

Then K' is a flat, Δ is a full-dimensional cone of $\mathcal{Q}|_H$, and there is an adapted complement B' to K' such that

$$B = H \oplus B', \qquad S = K' \oplus B'.$$

Proof. If $K = S$, take $B = 0$. Otherwise choose any linear section $s_0 : S/K \to S$. Since D_K is pointed, choose $f \in (S/K)^*$ positive on $\overline{D}_K \setminus \{0\}$. For $L > 0$, put

$$s_L(\bar{y}) = s_0(\bar{y}) + Lf(\bar{y})\xi.$$

This remains a section. Forms vanishing on K are unchanged, whereas

$$\ell(s_L(\bar{y})) = \ell(s_0(\bar{y})) + Lf(\bar{y})\ell(\xi) \qquad (\ell|_K \neq 0).$$

On the compact set $\overline{D}_K \cap \{\|\bar{y}\| = 1\}$, f has a positive minimum. The finitely many displayed expressions are therefore positive on $\overline{D}_K \cap \{\|\bar{y}\| = 1\}$ for all sufficiently large L. Since their signs are invariant under positive rescaling, they are positive on $\overline{D}_K \setminus \{0\}$. Thus $\operatorname{im} s_L$ is adapted.

For the second assertion, let $\mathcal{Z} = \{\ell \in \mathcal{N}_S : \ell|_K = 0,\ \ell|_\Delta = 0\}$. The face-span identity in B gives

$$H = B \cap \bigcap_{\ell \in \mathcal{Z}} \ker \ell, \qquad K' = \bigcap_{\ell \in \mathcal{Z}} \ker \ell \subseteq S.$$

Thus K' is a flat. Each form vanishing on K is positive or identically zero on Δ, and every other form is positive there by (39). Consequently Δ is a cone of $\mathcal{Q}|_H$, of full dimension in H.

If $K' = S$, take $B' = 0$. Otherwise fix $\theta \in \Delta$. If $\ell|_{K'} \neq 0$, then either $\ell|_K \neq 0$, or $\ell|_K = 0$ and $\ell|_H \neq 0$. In both cases $\ell(\theta) > 0$. Choose a section $s_0 : S/K' \to B$, which exists because $S = K \oplus B$ and $K' = K \oplus H$. Apply the preceding construction with K', $D_{K'}$, and $\xi = \theta$. Since both s_0 and θ lie in B, the resulting adapted complement B' lies in B and satisfies $B = H \oplus B'$. □

Definition 6.7. A tuple $(\mathscr{V}, K, \chi, B)$ is admissible for (S, C) if $K \subseteq S$ is a flat of $\mathcal{Q}|_S$, $\chi \in K^*$, B is an adapted complement to K in S, and the following conditions hold:

(i) $\mathscr{V} \subseteq \mathscr{P}$ is finite dimensional;

(ii) $T_X\mathscr{V} \subseteq \mathscr{V}$ for every $X \in K$;

(iii) every $v \in \mathscr{V}$ can be written as

$$v = \sum_{j=1}^{N} a_j v_j,$$

where v_j is a residue period on a face G_j satisfying

$$K \subseteq W_{G_j}, \qquad \ell_{G_j}|_K = \chi \qquad (1 \leq j \leq N).$$

Let Δ be a nonzero relatively open face of $\overline{C_{K,B}}$, put $H = \operatorname{span}\Delta$ and $K' = K \oplus H$, and fix an adapted $B' \subseteq B$ as in Lemma 6.6.

Lemma 6.8. Suppose $(\mathscr{V}, K, \chi, B)$ is admissible for (S, C). For every $\gamma \in H^*$, put

$$\mathscr{V}_\gamma := \operatorname{span}\{T_X \mathcal{C}_\gamma^{H,\Delta} v : X \in K',\ v \in \mathscr{V}\}.$$

The tuple

$$(\mathscr{V}_\gamma, K', \chi \oplus \gamma, B')$$

is also admissible for (S, C).

Proof. Lemma 6.6 gives the flat K', its adapted complement B', and the cone Δ of $\mathcal{Q}|_H$. Fix $\gamma \in H^*$. We first show that every $\mathcal{C}_\gamma^{H,\Delta} v$ is a finite linear combination of residue periods on faces F satisfying

$$K' \subseteq W_F, \qquad \ell_F|_{K'} = \chi \oplus \gamma.$$

Fix $v \in \mathscr{V}$ and $\theta \in \Delta$. By condition (iii) of Definition 6.7, write

$$v = \sum_j a_j v_j, \qquad K \subseteq W_{G_j}, \qquad \ell_{G_j}|_K = \chi,$$

where v_j is a residue period on G_j. Choose $T > \gamma(\theta)$. For each j with $\theta \notin W_{G_j}$, apply Theorem 3.1 with objective θ and threshold T; for $\theta \in W_{G_j}$, use the term v_j itself. Combining these expressions gives

$$v = \sum_p b_p u_p + \sum_\ell r_\ell.$$

Here u_p is a period on a face F_p with

$$F_p \subseteq \operatorname{aff}(G_{j(p)}), \qquad \theta \in W_{F_p},$$

and each remainder r_ℓ has a contour base c_ℓ satisfying $\theta(c_\ell) > T$.

For $Y \in K$, the equality $Y|_{G_{j(p)}} = \chi(Y)$ extends to $\operatorname{aff}(G_{j(p)})$ and then restricts to F_p. Hence

$$K \subseteq W_{F_p}, \qquad \ell_{F_p}|_K = \chi.$$

Since Δ is a cone of $\mathcal{Q}|_H$, Lemma A.1 applies to $\theta \in \Delta \cap W_{F_p}$ and gives

$$H = \operatorname{span}\Delta \subseteq W_{F_p}, \qquad K' = K \oplus H \subseteq W_{F_p}.$$

Put $\beta_p = \ell_{F_p}|_H$. By Lemma 5.1, u_p has H-character β_p and K'-character $\chi \oplus \beta_p$. Group the terms as

$$w_\beta = \sum_{p:\, \beta_p = \beta} b_p u_p, \qquad v = \sum_\beta w_\beta + \sum_\ell r_\ell, \qquad w_\beta \in \mathscr{P} \cap E_\beta^H.$$

Because $\theta(c_\ell) > T > \gamma(\theta)$, (38), with both subspaces equal to H and cone Δ, and the projection property of Lemma 6.5 give

$$\mathcal{C}_\gamma^{H,\Delta} v = \mathcal{C}_\gamma^{H,\Delta} w_\gamma = w_\gamma = \sum_{p:\, \beta_p = \gamma} b_p u_p.$$

For every p with $\beta_p = \gamma$, $\ell_{F_p}|_{K'} = \chi \oplus \gamma$, proving the claimed representation.

We now check the three conditions for $\mathscr{V}_\gamma$. For (i), choose a basis $v^{(1)},\dots,v^{(r)}$ of $\mathscr{V}$ and put $w_i=\mathcal{C}_\gamma^{H,\Delta}v^{(i)}$. The representation just proved and Lemma 5.1 make each map

$$K'\longrightarrow \mathcal{S}'_Q(\mathcal{F}'_\beta),\qquad X\longmapsto e^{(\chi\oplus\gamma)(X)}T_Xw_i$$

polynomial. Thus each K'-dilation span is finite dimensional, and so is

$$\mathscr{V}_\gamma=\sum_{i=1}^{r}\operatorname{span}\{T_Xw_i : X\in K'\}.$$

For (ii), $T_ZT_Xw_i=T_{Z+X}w_i$ for $X,Z\in K'$, so $T_Z\mathscr{V}_\gamma\subseteq\mathscr{V}_\gamma$. For (iii), (36) writes every T_Xu_p as a residue period on the same face F_p. Consequently every element of $\mathscr{V}_\gamma$ is a finite linear combination of periods on faces satisfying $K'\subseteq W_{F_p}$ and $\ell_{F_p}|_{K'}=\chi\oplus\gamma$. □

Lemma 6.9. Let $(\mathscr{V},K,\chi,B)$ be admissible for (S,C), with $K\subsetneq S$. Let Δ be a nonzero relatively open face of $\overline{C_{K,B}}$ and put $H=\operatorname{span}\Delta$. Then, for every $\alpha\in S^*$ and $v\in\mathscr{V}$,

$$\mathcal{C}_\alpha^{S,C}(\mathcal{C}_{\alpha|_H}^{H,\Delta}v)=\mathcal{C}_\alpha^{S,C}v. \tag{40}$$

Proof. By condition (iii) of Definition 6.7 and Lemma 5.1, $v\in E_\chi^K$. Since coefficient maps preserve characters by Lemma 6.5,

$$\mathcal{C}_\alpha^{S,C}v,\quad \mathcal{C}_\alpha^{S,C}(\mathcal{C}_{\alpha|_H}^{H,\Delta}v)\in E_\chi^K\cap E_{\alpha|_K}^K.$$

Lemma 6.3 gives

$$\alpha|_K\neq\chi\quad\Longrightarrow\quad E_\chi^K\cap E_{\alpha|_K}^K=\{0\},$$

so both sides vanish in this case.

Assume $\alpha|_K=\chi$. Fix $\theta\in\Delta$ and write $v=\sum_{j=1}^N a_jv_j$, with source faces G_j as in condition (iii). Choose

$$T>\max\Big(\{\alpha(\theta)\}\cup\{\ell_{G_j}(\theta):1\le j\le N,\ \theta\in W_{G_j}\}\Big).$$

Apply Theorem 3.1 to each v_j with objective θ and threshold T, using v_j itself when $\theta\in W_{G_j}$. Grouping by H-character as in the proof of Lemma 6.8 gives

$$v=\sum_{\gamma\in\Gamma}w_\gamma+\sum_\ell r_\ell,\qquad w_\gamma\in\mathscr{P}\cap E_\gamma^H,\qquad \gamma(\theta)\le T\quad(\gamma\in\Gamma),$$

where Γ is finite. Define $w_\gamma=0$ for $\gamma\notin\Gamma$. Choose remainder bases c_ℓ with $\theta(c_\ell)>T$. Formula (38), applied on H with cone Δ, yields

$$\mathcal{C}_\gamma^{H,\Delta}v=w_\gamma\qquad(\gamma\in\Gamma\cup\{\alpha|_H\}).$$

Adaptedness gives

$$\theta\in\Delta\subseteq\overline{C},\qquad\theta(c_\ell)>T>\alpha(\theta).$$

Apply (38) again, now on S with cone C and subspace H:

$$\mathcal{C}_\alpha^{S,C}v=\sum_{\substack{\gamma\in\Gamma\\ \gamma=\alpha|_H}}\mathcal{C}_\alpha^{S,C}w_\gamma=\mathcal{C}_\alpha^{S,C}w_{\alpha|_H}=\mathcal{C}_\alpha^{S,C}(\mathcal{C}_{\alpha|_H}^{H,\Delta}v).$$

□

6.5 Finite character families

Keep the geometric data (K, χ, H, Δ) above. A fixed source period map has a threshold expansion independent of its input, so its possible characters satisfy the following finiteness properties. At a boundary weight, several characters may share a value.

Corollary 6.10. Fix a source residue period map $R : \mathcal{A} \to \mathscr{P}$ on a face G with $K \subseteq W_G$ and $\ell_G|_K = \chi$, and put

$$\Gamma_\Delta(R) := \{\alpha \in H^* : \mathcal{C}_\alpha^{H,\Delta} R \neq 0\}.$$

For every $\xi \in \overline{\Delta}$ and $B \in \mathbb{R}$, the set

$$\{\alpha(\xi) : \alpha \in \Gamma_\Delta(R),\ \alpha(\xi) \leq B\}$$

is finite. Moreover, $\Gamma_\Delta(R)$ is locally finite, and for every nonempty compact $Q \subset \Delta$ and $B \in \mathbb{R}$, only finitely many $\alpha \in \Gamma_\Delta(R)$ satisfy $\alpha(Y) \leq B$ at some $Y \in Q$.

Proof. Fix $\xi \in \overline{\Delta}$ and $T > B$. If ξ is nonconstant on G, apply Theorem 3.1 to R; otherwise use R itself as the single term. In either case, write

$$R = \sum_{j=1}^{N} A_j + \sum_{\ell} R_\ell, \qquad \xi|_{F_j} = \lambda_j, \qquad \xi(c_\ell) > T,$$

where A_j is a residue period map on F_j and c_ℓ is a contour base for R_ℓ. These maps and faces are independent of the input. By Lemma 5.1, the image of A_j has character value λ_j along ξ. For $\alpha(\xi) \leq B$, formula (38), with cone Δ and subspace $\mathbb{R}\xi$, gives

$$\mathcal{C}_\alpha^{H,\Delta} R = \sum_{j:\, \lambda_j = \alpha(\xi)} \mathcal{C}_\alpha^{H,\Delta} A_j.$$

Thus $\alpha \in \Gamma_\Delta(R)$ and $\alpha(\xi) \leq B$ imply $\alpha(\xi) \in \{\lambda_1, \dots, \lambda_N\}$, proving the first assertion.

If $\xi \in \Delta$, the argument of Lemma 6.8 gives $H \subseteq W_{F_j}$. Each A_j then has the full H-character $\beta_j = \ell_{F_j}|_H$. By continuity, choose a neighborhood $N_\xi \subseteq \Delta$ such that

$$\vartheta(c_\ell) > B \qquad (\vartheta \in N_\xi,\ \text{every } \ell).$$

For $\alpha(\vartheta) \leq B$, apply (38) to the same expansion, now with subspace H and weight ϑ:

$$\mathcal{C}_\alpha^{H,\Delta} R = \sum_{j:\, \beta_j = \alpha} A_j.$$

Hence every nonzero character below B anywhere in N_ξ belongs to $\{\beta_1, \dots, \beta_N\}$. A finite cover of a compact $Q \subset \Delta$ gives the asserted finiteness over Q.

Finally, fix $\theta_0 \in \Delta$ and a compact set $A \subset H^*$. For nonempty A, put $B_A = \max_{\alpha \in A} \alpha(\theta_0)$. The assertion just proved, with $Q = \{\theta_0\}$, makes

$$A \cap \Gamma_\Delta(R) \subseteq \{\alpha \in \Gamma_\Delta(R) : \alpha(\theta_0) \leq B_A\}$$

finite. Hence $\Gamma_\Delta(R)$ is locally finite in H^*. □

For the next two results, fix admissible data $(\mathscr{V}, K, \chi, B)$ and write

$$\mathcal{C}_\alpha := \mathcal{C}_\alpha^{H,\Delta}|_{\mathscr{V}} : \mathscr{V} \longrightarrow \mathscr{V}_\alpha, \qquad \Gamma_\Delta := \{\alpha \in H^* : \mathcal{C}_\alpha \neq 0\}.$$

Choose residue-period expressions for a basis of $\mathscr{V}$. Their finitely many source maps show that Γ_Δ inherits the finiteness properties of Corollary 6.10. Use the order

$$\alpha \leq_\Delta \beta \quad \Longleftrightarrow \quad (\beta - \alpha)(Y) \geq 0 \quad (Y \in \Delta).$$

Corollary 6.11. There is a finite $A_\Delta \subseteq \Gamma_\Delta$ such that, for every $\beta \in H^*$,

$$\mathcal{C}_\beta v = \sum_{\substack{\alpha \in A_\Delta \\ \alpha \leq_\Delta \beta}} B_{\beta\alpha}(\mathcal{C}_\alpha v) \qquad (v \in \mathscr{V}). \tag{41}$$

Here

$$B_{\beta\alpha} : \mathcal{C}_\alpha(\mathscr{V}) \longrightarrow \mathcal{C}_\beta(\mathscr{V})$$

are linear maps. For $v \in \mathscr{V}$, put $A_\Delta(v) = \{\alpha \in A_\Delta : \mathcal{C}_\alpha v \neq 0\}$. If $A_\Delta(v) \neq \varnothing$, then

$$\min_{\beta:\, \mathcal{C}_\beta v \neq 0} \beta(Y) = \min_{\alpha \in A_\Delta(v)} \alpha(Y), \qquad Y \in \Delta.$$

Proof. Choose nonzero generators $\xi_1, \ldots, \xi_q$ of the polyhedral cone $\overline{\Delta}$, including opposite generators for any lineality. By Corollary 6.10, for each i the possible values $\alpha(\xi_i)$ have finite sublevel sets. Every infinite sequence of such values has an infinite nondecreasing subsequence: either a value repeats infinitely often, or one can successively choose strictly larger values. Passing successively to subsequences for $i = 1, \ldots, q$ therefore makes all coordinates nondecreasing. Since the ξ_i generate $\overline{\Delta}$, the resulting characters are nondecreasing for $\leq_\Delta$. Lemma A.2 applied to the nonzero maps then gives (41); for a zero map take all $B_{\beta\alpha} = 0$. If $\mathcal{C}_\beta v \neq 0$, the factorization supplies some $\alpha \in A_\Delta(v)$ with $\alpha \leq_\Delta \beta$. Since $A_\Delta(v)$ is itself a subset of the nonzero characters of v, the two minima agree. □

6.6 Relative estimates away from cone walls

For a finite set $\Lambda \subset S^*$, write

$$J_\Lambda^{S,C} := \sum_{\alpha \in \Lambda} \mathcal{C}_\alpha^{S,C} : \mathscr{P} \longrightarrow \mathscr{P}.$$

Let A_Δ be the set from Corollary 6.11 and fix a finite set $A_\Delta \subseteq \Lambda \subseteq \Gamma_\Delta$.

Lemma 6.12. There is a finite $\mathcal{K}_* \subset \mathcal{F}$, depending on $\mathscr{V}$ and A_Δ, such that the following holds for every bounded $\mathcal{K} \supseteq \mathcal{K}_*$. For each compact $\Theta \subset \Delta \cap \{\|Y\| = 1\}$, there is $\delta_\Theta > 0$, independent of $\mathcal{K}$, such that, for every m, there are $k \geq m$ and $C_m, A_m > 0$ with

$$\|T_{t\theta+\eta}(v - J_\Lambda^{H,\Delta} v)\|_{-k,\mathcal{K}} \leq C_m e^{-\delta_\Theta t + A_m\|\eta\|} \|T_{t\theta+\eta} v\|_{-m,\mathcal{K}} \tag{42}$$

for all $v \in \mathscr{V}$, $t \geq 1$, $\theta \in \Theta$, and $\eta \in W$.

Proof. For $\alpha \in A_\Delta$, put $\mathscr{G}_\alpha = \operatorname{span}\{T_Y \mathcal{C}_\alpha v : Y \in H,\ v \in \mathscr{V}\}$. These spaces are finite dimensional by Lemma 6.8. Choose $\mathcal{K}_*$ to contain the auxiliary tests in (32) for $\mathscr{V}$ and all these $\mathscr{G}_\alpha$. Fix bounded $\mathcal{K} \supseteq \mathcal{K}_*$, m, and $\theta_0 \in \Theta$. We first prove the estimate for $\eta = 0$ and θ near θ_0.

The contour bounds use a fixed norm on $\mathscr{V}$, whereas the desired estimate uses $\|T_{t\theta}v\|_{-m,\mathcal{K}}$. To compare them, use the detecting pairs (h_j, f_j) chosen above, with $f_j \in \mathcal{K}_*$, to define the norm in (32). Write $v = T_{-t\theta}T_{t\theta}v$, move the inverse dilation to each h_j, and apply (33). This gives

$$\|v\|_{\mathscr{V}} \le Ce^{N_m t}\|T_{t\theta}v\|_{-m,\mathcal{K}} \qquad (\theta \in \Theta). \tag{43}$$

By (43), a contour rate greater than N_m gives relative decay. We therefore choose an auxiliary threshold expansion after fixing m, while keeping $J_\Lambda^{H,\Delta}v$ fixed.

Let G_j be the faces of the residue periods used to express the chosen basis of $\mathscr{V}$. Choose a threshold

$$T > \max\big(\{N_m + 2\} \cup \{\alpha(\theta_0) : \alpha \in \Lambda\} \cup \{\theta_0|_{G_j} : \theta_0 \text{ is constant on } G_j\}\big).$$

Apply Theorem 3.1 to those periods with objective θ_0 and threshold T. Each remainder face has LP value greater than T, so choose a contour base c_ℓ in each such face with $\theta_0(c_\ell) > T$. Lemmas 6.5 and 6.8 identify the character groups with the canonical maps $\mathcal{C}_\beta$. By linearity, the expansion is

$$v = q + r, \qquad q = \sum_{\beta\in\Gamma} \mathcal{C}_\beta v, \qquad \Lambda \subseteq \Gamma \subseteq \Gamma_\Delta,$$

for every $v \in \mathscr{V}$, where Γ is finite and independent of v, and r is the sum of the periods based at the c_ℓ. We estimate the two terms in

$$v - J_\Lambda^{H,\Delta}v = (q - J_\Lambda^{H,\Delta}v) + r.$$

Since the bases c_ℓ are finite in number, they satisfy $\theta(c_\ell) > N_m + 3/2$ throughout a sufficiently small neighborhood of θ_0 in Θ. Absorbing the polynomial factors in the contour bounds of Lemma 5.1 into this gap, and then using (43), gives for some $k \ge m$

$$\|T_{t\theta}r\|_{-k,\mathcal{K}} \le Ce^{-(N_m+1)t}\|v\|_{\mathscr{V}} \le C'e^{-t}\|T_{t\theta}v\|_{-m,\mathcal{K}}.$$

If $A_\Delta = \varnothing$, (41) gives $\Gamma_\Delta = \varnothing$ and $q = J_\Lambda^{H,\Delta}v = 0$. The remainder estimate already proves the local bound with $\delta_\Theta = 1/2$.

It remains to control $q - J_\Lambda^{H,\Delta}v$. For each $\beta \in \Gamma$, let

$$\mathscr{G}_\beta = \operatorname{span}\{T_Y\mathcal{C}_\beta v : Y \in H,\ v \in \mathscr{V}\}$$

and fix a norm on this finite-dimensional space. Increase k so that $\|\cdot\|_{-k,\mathcal{K}}$ is finite on every $\mathscr{G}_\beta$. The matrices of $e^{\beta(Y)}T_Y$ and its inverse on $\mathscr{G}_\beta$ are polynomial in Y. Hence

$$\|T_{\pm t\theta}w\|_{\mathscr{G}_\beta} \le C(1+t)^D e^{\mp t\beta(\theta)}\|w\|_{\mathscr{G}_\beta}.$$

For every β, finite dimension gives $\|w\|_{-k,\mathcal{K}} \le C_\beta\|w\|_{\mathscr{G}_\beta}$. For $\alpha \in A_\Delta$ the reverse estimate also holds, because $\mathcal{K}_*$ contains a detecting set for $\mathscr{G}_\alpha$. Apply the factorization (41) in these fixed norms, the forward dilation bound for β, and the inverse bound for α. For $\beta \in \Gamma \setminus \Lambda$ this yields

$$\|T_{t\theta}\mathcal{C}_\beta v\|_{-k,\mathcal{K}} \le C(1+t)^{2D} \sum_{\substack{\alpha\in A_\Delta \\ \alpha \le_\Delta \beta}} e^{-t(\beta-\alpha)(\theta)}\|T_{t\theta}\mathcal{C}_\alpha v\|_{-k,\mathcal{K}}.$$

Only the spaces indexed by A_Δ require detecting tests; the additional characters in Γ do not change $\mathcal{K}_*$. The exponent differences in this sum have a positive lower bound independent of m and θ_0. Indeed, for $\alpha \in A_\Delta$ and $\beta \in \Gamma_\Delta \setminus \Lambda$ with $\alpha \le_\Delta \beta$, compactness of Θ inside Δ gives

$$(\beta - \alpha)(\theta) \ge c_\Theta\|\beta - \alpha\| \qquad (\theta \in \Theta).$$

If there are no such pairs, take $\varepsilon_\Theta = 1$. Otherwise, the local finiteness of Γ_Δ established above from Corollary 6.10 gives, for each $\alpha \in A_\Delta$, a number $d_\alpha > 0$ such that $B(\alpha, d_\alpha) \cap \Gamma_\Delta = \{\alpha\}$. Since A_Δ is finite, $d := \min_{\alpha \in A_\Delta} d_\alpha > 0$. For the pairs above, $\beta \neq \alpha$, so

$$\|\beta - \alpha\| \geq d, \qquad (\beta - \alpha)(\theta) \geq c_\Theta d =: \varepsilon_\Theta > 0.$$

Set $\delta_\Theta = \frac{1}{2}\min\{1, \varepsilon_\Theta\}$.

Lemma 6.3 directly bounds the character components of q in the same order:

$$\|T_{t\theta}\mathcal{C}_\alpha v\|_{-k,\mathcal{K}} = \|\mathsf{P}_\alpha T_{t\theta} q\|_{-k,\mathcal{K}} \leq C\|T_{t\theta} q\|_{-k,\mathcal{K}} \qquad (\alpha \in A_\Delta).$$

Summing the preceding coefficient bounds over $\beta \in \Gamma \setminus \Lambda$ and using the gap gives

$$\|T_{t\theta}(q - J_\Lambda^{H,\Delta} v)\|_{-k,\mathcal{K}} \leq C(1+t)^{2D} e^{-\varepsilon_\Theta t}\|T_{t\theta} q\|_{-k,\mathcal{K}} \leq C' e^{-\delta_\Theta t}\|T_{t\theta} q\|_{-k,\mathcal{K}},$$

where $\delta_\Theta < \varepsilon_\Theta$ absorbs the polynomial factor. Since $q = v - r$ and $k \geq m$, the remainder bound gives

$$\|T_{t\theta} q\|_{-k,\mathcal{K}} \leq (1 + Ce^{-t})\|T_{t\theta} v\|_{-m,\mathcal{K}}.$$

Using $v - J_\Lambda^{H,\Delta} v = (q - J_\Lambda^{H,\Delta} v) + r$ and $\delta_\Theta \leq 1$ now yields

$$\|T_{t\theta}(v - J_\Lambda^{H,\Delta} v)\|_{-k,\mathcal{K}} \leq Ce^{-\delta_\Theta t}\|T_{t\theta} v\|_{-m,\mathcal{K}}.$$

This proves the estimate for $\eta = 0$ and θ near θ_0. In both cases, $\delta_\Theta > 0$ is independent of m and θ_0. For fixed m, choose finitely many of the neighborhoods above covering Θ, with corresponding orders $k_1, \ldots, k_N$ and constants $C_1, \ldots, C_N$. Set $k = \max_i k_i$ and $C = \max_i C_i$. Since $\|\cdot\|_{-k,\mathcal{K}} \leq \|\cdot\|_{-k_i,\mathcal{K}}$, we obtain

$$\|T_{t\theta}(v - J_\Lambda^{H,\Delta} v)\|_{-k,\mathcal{K}} \leq Ce^{-\delta_\Theta t}\|T_{t\theta} v\|_{-m,\mathcal{K}} \qquad (t \geq 1,\ \theta \in \Theta).$$

For arbitrary $\eta \in W$, apply (33) to T_η at order k and to $T_{-\eta}$ at order m. This introduces the factor $C_k C_m e^{(a_k + a_m)\|\eta\|}$ and proves (42) with $A_m = a_k + a_m$. □

6.7 Finite coefficient sums near cone walls

For $K \subsetneq S$ and $y \in C_{K,B}$, put

$$\nu_K(y) = \min_{\substack{\ell \in \mathcal{N}_S \\ \ell|_K = 0}} \ell(y).$$

In particular, $C_{0,S} = C$ and $\nu_0(y) = \min_{\ell \in \mathcal{N}_S} \ell(y)$.

We now return to the original cone $C \subset S$. Near a wall, we first extract coefficients along a face of the cone, then estimate the remaining dilation in a nested adapted complement.

Proposition 6.13. For every finite-dimensional $\mathscr{V}_0 \subseteq \mathscr{P}$ there is a finite set

$$\Lambda_0 \subseteq \{\alpha : \mathcal{C}_\alpha^{S,C}|_{\mathscr{V}_0} \neq 0\}$$

such that, for every finite set

$$\Lambda_0 \subseteq \Lambda \subseteq \{\alpha : \mathcal{C}_\alpha^{S,C}|_{\mathscr{V}_0} \neq 0\}$$

there is a finite $\mathcal{K}_* \subset \mathcal{F}$ such that, for every bounded $\mathcal{K} \supseteq \mathcal{K}_*$ and every m, there are $k \geq m$ and $M, \delta, R > 0$ with

$$\|T_y(v - J_\Lambda^{S,C} v)\|_{-k,\mathcal{K}} \leq Me^{-\delta\nu_0(y)}\|T_y v\|_{-m,\mathcal{K}} \qquad (v \in \mathscr{V}_0,\ y \in C,\ \nu_0(y) \geq R). \tag{44}$$

Proof. We use two claims: a finite family of sums suffices to approximate in every direction, and an intermediate coefficient sum inherits the estimates of two such approximations. Write $J_A = J_A^{S,C}$ throughout.

Claim 1 (finite approximation families). Given any finite set G of characters whose coefficient maps are nonzero on $\mathscr{V}_0$, there is a finite family $\mathfrak{A}_G$ of finite sets of such characters, each containing G, with the following property. There is a finite auxiliary test set $\mathcal{K}_G \subset \mathcal{F}$ such that, for every bounded $\mathcal{K} \supseteq \mathcal{K}_G$ and every m, there are $k \geq m$ and $M, \delta, R > 0$ such that, for every $y \in C$ with $\nu_0(y) \geq R$, some $A \in \mathfrak{A}_G$ satisfies

$$\|T_y(v - J_A v)\|_{-k,\mathcal{K}} \leq M e^{-\delta \nu_0(y)} \|T_y v\|_{-m,\mathcal{K}} \qquad (v \in \mathscr{V}_0).$$

To construct the family, start with the admissible tuple $(\mathscr{V}_0, 0, 0, S)$. For each nonzero relatively open face Δ of $\overline{C_{K,B}}$, put $H = \operatorname{span} \Delta$ and fix an adapted complement $B'_\Delta \subseteq B$ to $K \oplus H$ as in Lemma 6.6. At an input $(\mathscr{V}, K, \chi, B)$, set

$$\Lambda_\Delta = A_\Delta \cup \{\alpha|_H : \alpha \in G,\ \alpha|_K = \chi,\ \mathcal{C}^{H,\Delta}_{\alpha|_H}|_{\mathscr{V}} \neq 0\},$$

where A_Δ is the finite determining set from Corollary 6.11. The maps $\mathcal{C}^{H,\Delta}_\gamma$, $\gamma \in \Lambda_\Delta$, give the output spaces $\mathscr{V}_\gamma$ of Lemma 6.8. That lemma makes each output tuple

$$(\mathscr{V}_\gamma, K \oplus H, \chi \oplus \gamma, B'_\Delta)$$

admissible. Repeat on every output, stopping at $K = S$. Each recursive choice of faces defines a rooted tree whose vertices are admissible tuples and whose edges are the selected coefficient maps: at a vertex with chosen face Δ, there is one outgoing edge for each $\gamma \in \Lambda_\Delta$. Every branch has length at most $\dim S$, since $\dim K$ strictly increases along each edge. The collection of trees obtained by varying the face choices is therefore finite, since each vertex admits only finitely many face choices and each choice has finitely many outputs. For each input tuple $\mathfrak{v} = (\mathscr{V}, K, \chi, B)$ occurring in this construction, let $\mathfrak{T}(\mathfrak{v})$ be the finite family of trees rooted at that tuple. For $T \in \mathfrak{T}(\mathfrak{v})$, let $\mathcal{B}(T)$ be its branches from the root to vertices with $K = S$. If $b \in \mathcal{B}(T)$ has successive edge maps $Q_{b,1}, \ldots, Q_{b,r_b}$, define

$$L_T := \sum_{b \in \mathcal{B}(T)} Q_{b,r_b} \circ \cdots \circ Q_{b,1} : \mathscr{V} \longrightarrow \mathscr{P}.$$

The empty composition is the identity. Thus at $K = S$, the tree consists of a single vertex and $L_T = I$.

Consider a branch reaching $K = S$, and denote its successive edge maps by $Q_i = \mathcal{C}^{H_i,\Delta_i}_{\gamma_i}$, $1 \leq i \leq r$. Its terminal tuple has the form $(\mathscr{V}_*, S, \alpha, 0)$, where $\alpha|_{H_i} = \gamma_i$. For $v \in \mathscr{V}_0$, put $v_0 = v$ and $v_i = Q_i v_{i-1}$. Admissibility gives $v_r \in E^S_\alpha$, so Lemma 6.5 gives $\mathcal{C}^{S,C}_\alpha v_r = v_r$. At every preceding step, (40) gives $\mathcal{C}^{S,C}_\alpha v_i = \mathcal{C}^{S,C}_\alpha v_{i-1}$. Consequently

$$Q_r \cdots Q_1 v = v_r = \mathcal{C}^{S,C}_\alpha v_r = \mathcal{C}^{S,C}_\alpha v_0 = \mathcal{C}^{S,C}_\alpha v.$$

Thus the composite along this branch is exactly the original α-coefficient map. Within a fixed tree, two distinct branches first separate at edges with different characters $\gamma \neq \gamma'$ on the same subspace H. Their terminal characters restrict to γ and γ' on H, respectively, and hence are distinct. Summing the branch composites within a fixed tree gives $J_A^{S,C}$, where A is that tree's set of terminal characters, each occurring once. Discard terminal characters whose maps vanish on $\mathscr{V}_0$. Every resulting set contains G: by (40), a nonzero α-coefficient remains nonzero after extraction at $\alpha|_H$, so it survives to a terminal branch. These sets form $\mathfrak{A}_G$.

For every tuple and every face choice in this finite collection of trees, take the finite detecting set of Lemma 6.12. Their union is a finite set $\mathcal{K}_G$ independent of the test orders. Fix bounded $\mathcal{K} \supseteq \mathcal{K}_G$ for the estimates that follow.

We prove the following statement by induction on $\dim S - \dim K$: for every input tuple $\mathfrak{v} = (\mathscr{V}, K, \chi, B)$ in the construction with $K \neq S$ and every m, there are $k \geq m$ and $M, \delta, R > 0$ such that, for each $y \in C_{K,B}$ with $\nu_K(y) \geq R$, some $T \in \mathfrak{T}(\mathfrak{v})$ satisfies

$$\|T_y(v - L_T v)\|_{-k,\mathcal{K}} \leq M e^{-\delta \nu_K(y)} \|T_y v\|_{-m,\mathcal{K}} \qquad (v \in \mathscr{V}).$$

The base case $K = S$ has $L_T = I$ and zero error. For the induction step, fix m and a unit direction in a relatively open face Δ of $\overline{C_{K,B}}$, and put $H = \operatorname{span} \Delta$. Write $B' = B'_\Delta$, and let $P_\Delta : B \to H$ be the projection along B' in the decomposition $B = H \oplus B'$. For $y/\|y\|$ near this direction, put

$$t = \|P_\Delta y\|, \qquad \theta = P_\Delta y/t, \qquad \eta = (I - P_\Delta)y \in B', \qquad y = t\theta + \eta.$$

Locally, θ stays in a compact subset of the unit directions of Δ, $t \asymp \|y\|$, and $\|\eta\|/t$ can be made arbitrarily small. Fix that compact subset and then shrink the neighborhood so that the constants of Lemma 6.12 satisfy $A_m \|\eta\| \leq \delta_0 t/2$. The lemma gives, with $q = \sum_{\gamma \in \Lambda_\Delta} \mathcal{C}_\gamma^{H,\Delta} v$,

$$\|T_y(v - q)\|_{-l,\mathcal{K}} \leq M e^{-\delta t} \|T_y v\|_{-m,\mathcal{K}}$$

for some $l \geq m$ and $\delta = \delta_0/2$. Since $q = v - (v - q)$ and $l \geq m$, the triangle inequality gives

$$\|T_y q\|_{-l,\mathcal{K}} \leq \|T_y v\|_{-l,\mathcal{K}} + \|T_y(v - q)\|_{-l,\mathcal{K}} \leq (1 + M e^{-\delta t}) \|T_y v\|_{-m,\mathcal{K}} \leq (1 + M) \|T_y v\|_{-m,\mathcal{K}}.$$

Lemma 6.3, on the W-dilation span of the selected coefficient images of a basis of $\mathscr{V}$, therefore gives

$$\|T_y \mathcal{C}_\gamma^{H,\Delta} v\|_{-l,\mathcal{K}} \leq M' \|T_y v\|_{-m,\mathcal{K}},$$

uniformly for the selected γ.

For a terminal output, let $L_\gamma = I$. For a nonterminal output, put $K' = K \oplus H$ and $z = T_{t\theta} \mathcal{C}_\gamma^{H,\Delta} v \in \mathscr{V}_\gamma$. Its dilated contribution is $T_\eta z = T_y \mathcal{C}_\gamma^{H,\Delta} v$. If $\ell|_{K'} = 0$, then $\ell(\eta) = \ell(y) > 0$. These forms are a subset of those vanishing on K, so

$$\eta \in C_{K',B'}, \qquad \nu_{K'}(\eta) \geq \nu_K(y).$$

The output tuple $\mathfrak{v}_\gamma = (\mathscr{V}_\gamma, K', \chi \oplus \gamma, B')$ has smaller remaining dimension, since $H \neq 0$. The induction hypothesis at $\mathfrak{v}_\gamma$, with input order l and dilation parameter η, supplies a tree $T_\gamma \in \mathfrak{T}(\mathfrak{v}_\gamma)$ with sum $L_\gamma = L_{T_\gamma}$ and an output order $k_\gamma \geq l$ such that

$$\begin{aligned} \|T_\eta(I - L_\gamma)z\|_{-k_\gamma,\mathcal{K}} &\leq M_\gamma e^{-\delta_\gamma \nu_{K'}(\eta)} \|T_\eta z\|_{-l,\mathcal{K}} \\ &\leq M_\gamma M' e^{-\delta_\gamma \nu_K(y)} \|T_y v\|_{-m,\mathcal{K}}. \end{aligned}$$

The second inequality uses $\nu_{K'}(\eta) \geq \nu_K(y)$ and the coefficient estimate above, since $T_\eta z = T_y \mathcal{C}_\gamma^{H,\Delta} v$.

Attaching these trees to the edges indexed by $\gamma \in \Lambda_\Delta$ gives a tree $T \in \mathfrak{T}(\mathfrak{v})$: the construction allows every face choice at every subsequent input. Its sum is

$$L_T v = \sum_{\gamma \in \Lambda_\Delta} L_\gamma \mathcal{C}_\gamma^{H,\Delta} v.$$

Subtracting this from v separates the first extraction error $v-q$ from the errors in approximating the individual coefficients. Each L_γ commutes with $T_{t\theta}$ by Lemma 6.5, and $T_y = T_\eta T_{t\theta}$, so

$$T_y(v - L_T v) = T_y(v-q) + \sum_{\gamma\in\Lambda_\Delta} T_\eta (I - L_\gamma) T_{t\theta} \mathcal{C}_\gamma^{H,\Delta} v.$$

The summands are exactly the errors just estimated. For the first error, the finitely many wall forms give $\nu_K(y) \le L\|y\|$, while $t \asymp \|y\|$ gives $t \ge c\nu_K(y)$. Hence its bound $Me^{-\delta t}\|T_y v\|_{-m,\mathcal{K}}$ is at most $Me^{-c\delta\nu_K(y)}\|T_y v\|_{-m,\mathcal{K}}$. Choose k at least as large as l and every k_γ, and take the minimum of the positive decay rates above. For sufficiently large $\nu_K(y)$, all these estimates apply. Summing them in $\|\cdot\|_{-k,\mathcal{K}}$ gives the required bound for $T_y(v - L_T v)$. A finite cover of the compact set $\overline{C_{K,B}} \cap \{\|y\| = 1\}$ makes the estimate uniform: take the maximum of the local orders, constants, and thresholds, and the minimum of the positive local decay rates. This completes the induction. At the original tuple $(\mathscr{V}_0, 0, 0, S)$, these are the complete trees constructed above, and their sums are J_A with $A \in \mathfrak{A}_G$. This proves Claim 1.

Claim 2 (intermediate sums). For finite character sets $A \subseteq \Lambda \subseteq B$, every bounded $\mathcal{K} \subset \mathcal{F}$, and every k, there is $C > 0$ such that

$$\|T_y(v - J_\Lambda v)\|_{-k,\mathcal{K}} \le C\big(\|T_y(v - J_A v)\|_{-k,\mathcal{K}} + \|T_y(v - J_B v)\|_{-k,\mathcal{K}}\big) \qquad (v \in \mathscr{V}_0).$$

Indeed, set $\mathsf{P} = \sum_{\alpha\in\Lambda\setminus A} \mathsf{P}_\alpha$. It commutes with dilation and satisfies $\mathsf{P}(J_B - J_A)v = (J_\Lambda - J_A)v$. Thus

$$T_y(v - J_\Lambda v) = T_y(v - J_A v) - \mathsf{P} T_y (J_B - J_A) v.$$

Let $v_1, \dots, v_N$ be a basis of $\mathscr{V}_0$ and set

$$\mathscr{M} = \operatorname{span}\{T_X \mathcal{C}_\alpha^{S,C} v_j : X \in W,\ \alpha \in B,\ 1 \le j \le N\}.$$

Lemma 6.3 bounds each character projection on $\mathscr{M}$, and hence their finite sum P satisfies $\|\mathsf{P}w\|_{-k,\mathcal{K}} \le C_P\|w\|_{-k,\mathcal{K}}$ for $w \in \mathscr{M}$. Since $T_y(J_B - J_A)v \in \mathscr{M}$, the preceding identity gives

$$\begin{aligned}\|T_y(v - J_\Lambda v)\|_{-k,\mathcal{K}} &\le \|T_y(v - J_A v)\|_{-k,\mathcal{K}} + C_P\|T_y(J_B - J_A)v\|_{-k,\mathcal{K}} \\ &\le C\big(\|T_y(v - J_A v)\|_{-k,\mathcal{K}} + \|T_y(v - J_B v)\|_{-k,\mathcal{K}}\big),\end{aligned}$$

where the last step uses $(J_B - J_A)v = (v - J_A v) - (v - J_B v)$ and absorbs the projection bound into C.

We now prove the proposition. Apply Claim 1 with $G = \varnothing$ and define the finite set

$$\Lambda_0 = \bigcup_{A\in\mathfrak{A}_\varnothing} A.$$

Fix a finite $\Lambda \supseteq \Lambda_0$ as in the proposition. Claim 1 with $G = \Lambda$ gives a second finite family $\mathfrak{A}_\Lambda$, whose members all contain Λ. Thus every pair $A \in \mathfrak{A}_\varnothing$, $B \in \mathfrak{A}_\Lambda$ satisfies

$$A \subseteq \Lambda_0 \subseteq \Lambda \subseteq B.$$

Set $\mathcal{K}_* := \mathcal{K}_\varnothing \cup \mathcal{K}_\Lambda$ and fix bounded $\mathcal{K} \supseteq \mathcal{K}_*$ and an input order m. Choose a common output order $k \ge m$ and threshold R for the two applications of Claim 1, and take the smaller decay rate δ. Since both families are finite, Claim 2 has a common constant C for all their pairs at order k.

Now fix $y \in C$ with $\nu_0(y) \ge R$. Claim 1 selects $A \in \mathfrak{A}_\varnothing$ and $B \in \mathfrak{A}_\Lambda$ such that, for every $v \in \mathscr{V}_0$,

$$\begin{aligned}\|T_y(v - J_A v)\|_{-k,\mathcal{K}} &\le M_1 e^{-\delta\nu_0(y)}\|T_y v\|_{-m,\mathcal{K}}, \\ \|T_y(v - J_B v)\|_{-k,\mathcal{K}} &\le M_2 e^{-\delta\nu_0(y)}\|T_y v\|_{-m,\mathcal{K}}.\end{aligned}$$

Since $A \subseteq \Lambda \subseteq B$, Claim 2 gives

$$\begin{aligned}\|T_y(v - J_\Lambda v)\|_{-k,\mathcal{K}} &\le C\big(\|T_y(v - J_A v)\|_{-k,\mathcal{K}} + \|T_y(v - J_B v)\|_{-k,\mathcal{K}}\big)\\ &\le C(M_1 + M_2)e^{-\delta\nu_0(y)}\|T_y v\|_{-m,\mathcal{K}}.\end{aligned}$$

Setting $M = C(M_1 + M_2)$ proves (44). □

6.8 Construction of the chamber operators

Fix $S \in \mathcal{S}_0 \setminus \{0\}$ and a full-dimensional cone C of the restriction of $\mathcal{Q}$ to S. Apply the maps of Lemma 6.5 to EU. Thus the coefficient map on original inputs is the composition

$$\mathcal{C}_\alpha^{S,C} E : \mathcal{A} \longrightarrow \mathscr{P} \cap E_\alpha^S.$$

Put

$$\Gamma_{S,C} = \{\alpha \in S^* : \mathcal{C}_\alpha^{S,C} E \neq 0\}, \qquad H_\alpha = \{(\vartheta, r) \in C \times \mathbb{R} : r = \alpha(\vartheta)\}.$$

Let D range over the connected components of

$$(C \times \mathbb{R}) \setminus \bigcup_{\alpha \in \Gamma_{S,C}} H_\alpha.$$

For $(\vartheta, r) \in D$, put

$$\Gamma_{S,C,D} = \{\alpha \in \Gamma_{S,C} : \alpha(\vartheta) < r\}.$$

Define $\mathcal{L}_{S,C,D}, \mathcal{R}_{S,C,D} : \mathcal{A} \to \mathscr{P}$ by

$$\mathcal{L}_{S,C,D} = \sum_{\alpha \in \Gamma_{S,C,D}} \mathcal{C}_\alpha^{S,C} E, \qquad \mathcal{R}_{S,C,D} = E - \mathcal{L}_{S,C,D} \tag{45}$$

as justified by the following proposition.

Proposition 6.14. The walls H_α are locally finite. Each $\Gamma_{S,C,D}$ is finite and independent of the chosen point in D. The operators satisfy

$$\mathcal{L}_{S,C,D} + \mathcal{R}_{S,C,D} = E, \qquad \mathcal{L}_{S,C,D} E = \mathcal{L}_{S,C,D}, \qquad \mathcal{R}_{S,C,D} E = \mathcal{R}_{S,C,D}.$$

There are finitely many residue period maps $L_v, R_j : \mathcal{A} \to \mathscr{P}$ on faces $F_v \in \mathcal{I}_S$ and $G_j \notin \mathcal{I}_S$, respectively, such that

$$\mathcal{L}_{S,C,D} = \sum_v L_v, \qquad \mathcal{R}_{S,C,D} = \sum_j R_j.$$

Proof. Let $Q \subset C \times \mathbb{R}$ be compact, let Q_C be its projection to C, and choose $b \in \mathbb{R}$ such that $r \le b$ for every $(\vartheta, r) \in Q$. If H_α meets Q, then $\alpha(\vartheta) = r \le b$ for some $\vartheta \in Q_C$. Corollary 6.10, applied to the initial period map E, allows only finitely many such α. Thus the walls are locally finite. The same corollary shows that $\Gamma_{S,C,D}$ is finite at each point $(\vartheta, r) \in D$. For every $\alpha \in \Gamma_{S,C}$, the continuous function $r - \alpha(\vartheta)$ is nonzero on the connected set D, so its sign is constant there. Hence $\Gamma_{S,C,D}$ is independent of the chosen point.

Fix $(\vartheta, r) \in D$ and apply Theorem 3.1 to the initial period on A, with objective ϑ and threshold r. As an identity of maps on $\mathcal{A}$, this gives

$$E = \sum_p L_p + \sum_j R_j.$$

Each L_p is a residue period map on a face F_p on which ϑ is constant, with value at most r. Thus $\vartheta \in C \cap W_{F_p}$, and Lemma A.1 gives $S = \operatorname{span} C \subseteq W_{F_p}$. In particular, $F_p \in \mathcal{I}_S$. Put $\alpha_p = \ell_{F_p}|_S$. The image of L_p has S-character α_p, with $\alpha_p(\vartheta) \le r$.

Each remainder R_j has a contour base c_j satisfying $\vartheta(c_j) > r$. Therefore (38), with $K = S$, gives

$$\mathcal{C}_\alpha^{S,C} E = \sum_{p:\, \alpha_p = \alpha} L_p \qquad (\alpha(\vartheta) \le r),$$

where Lemma 6.5 makes $\mathcal{C}_\alpha^{S,C}$ the identity on the matching terms. If $\alpha(\vartheta) = r$, then $\mathcal{C}_\alpha^{S,C} E = 0$: otherwise $\alpha \in \Gamma_{S,C}$ and $(\vartheta, r) \in H_\alpha$, contrary to $(\vartheta, r) \in D$. Summing the displayed identity over the characters of the residue terms consequently gives

$$\sum_p L_p = \sum_{\alpha \in \Gamma_{S,C,D}} \mathcal{C}_\alpha^{S,C} E = \mathcal{L}_{S,C,D}.$$

Each R_j is a period on a chamber G_j that is full dimensional in A. Since ϑ is nonconstant on A, it is nonconstant on G_j, so $\vartheta \notin W_{G_j}$. As $\vartheta \in S$, this implies $S \not\subseteq W_{G_j}$, or $G_j \notin \mathcal{I}_S$. The contour expansion now gives

$$\sum_j R_j = E - \mathcal{L}_{S,C,D} = \mathcal{R}_{S,C,D}.$$

Finally, abbreviate the two operators by $\mathcal{L}$ and $\mathcal{R}$. Their definitions give $\mathcal{L} + \mathcal{R} = E$. Using $E^2 = E$ from Proposition 4.1, we obtain

$$\mathcal{L}E = \sum_{\alpha \in \Gamma_{S,C,D}} \mathcal{C}_\alpha^{S,C} E^2 = \mathcal{L}, \qquad \mathcal{R}E = (E - \mathcal{L})E = E - \mathcal{L} = \mathcal{R}.$$

□

Theorem 6.15 (Remainders at bounded normalizations)**.** There is a constant $c > 0$. For every $U \in \mathcal{A}$, one chamber $D_{S,C}$ can be chosen for each pair (S, C) so that the finite family

$$\{(\mathcal{L}_{S,C,D_{S,C}}, \mathcal{R}_{S,C,D_{S,C}})\}_{S,C}$$

satisfies the following estimate. For each pair (S, C) there is a finite $\mathcal{K}_* \subset \mathcal{F}$ such that, for every bounded $\mathcal{K} \supseteq \mathcal{K}_*$ and every sufficiently large m, there are $k \ge m$ and $C_m, A_m, \delta_m, R_m > 0$, allowed to depend on $\mathcal{K}$, such that

$$\|T_\omega \mathcal{R}_{S,C,D_{S,C}} U\|_{-k,\mathcal{K}} \le C_m e^{A_m \|b\| - \delta_m \mu_S(\omega)} \|T_\omega E U\|_{-m,\mathcal{K}} \tag{46}$$

whenever

$$p_S \omega \in C, \qquad b = b_S \omega, \qquad \mu_S(\omega) \ge c\|b\| + R_m.$$

Every net $(\omega_i)_{i \in I}$ with $\|\omega_i\| \to \infty$ has a subnet and a pair (S, C) for which, for every $a_i > 0$,

$$\boxed{\begin{array}{c} \{T_{\omega_i} E U / a_i\} \text{ bounded in } \mathcal{S}'_Q(\mathcal{F}'_\beta) \\ \Longrightarrow \quad \dfrac{T_{\omega_i} \mathcal{R}_{S,C,D_{S,C}} U}{a_i} \longrightarrow 0 \quad \text{in } \mathcal{S}'_Q(\mathcal{F}'_\beta). \end{array}} \tag{47}$$

Proof. Fix U, S, and C, and apply Proposition 6.13 to $\mathscr{V}_0 = \operatorname{span}\{EU\}$. Choose $\vartheta_C \in C$ and r_C above $\alpha(\vartheta_C)$ for every $\alpha \in \Lambda_0$, avoiding the walls H_α. Let $D_{S,C}$ contain (ϑ_C, r_C). The finite set

$$\Lambda = \{\alpha \in \Gamma_{S,C,D_{S,C}} : \mathcal{C}_\alpha^{S,C} EU \neq 0\}$$

contains Λ_0, and $\mathcal{L}_{S,C,D_{S,C}} U = J_\Lambda^{S,C} EU$. Fix the finite set $\mathcal{K}_*$ supplied by that proposition. The chamber choice precedes, and is independent of, all auxiliary test sets and Schwartz orders.

On C, $\nu_0 \asymp \mu_S$. Indeed, each proper normal span not containing S extends to a polar-normal hyperplane still not containing S, and each such hyperplane is W_F for a line face. Finitely many normal types and norm equivalence give the comparison. Thus (44) bounds the remainder at every $Y \in C$ with $\mu_S(Y)$ sufficiently large.

Write $\omega = Y + b$ with $Y = p_S\omega$. Choose $c > 0$, uniformly over the finitely many S, so that $\mu_S(Y) \geq \mu_S(\omega) - c\|b\|$. This constant depends only on the geometric data, and the hypothesis in (46) ensures the required lower bound. Apply T_b to the estimate and use T_{-b} to compare its input norm with that at ω. Equation (33) gives (46), uniformly over the finitely many (S, C).

For a net with $\|\omega_i\| \to \infty$, Lemma 6.1 gives a subnet with $b_i \to b$ and $\mu_S(\omega_i) \to \infty$. Restrict to a tail so that (b_i) is bounded. Then $\mu_S(p_S\omega_i) \to \infty$, so the projected weights eventually avoid the cone walls. Since there are finitely many cones, a further subnet lies in a fixed full-dimensional cone C. Fix an arbitrary bounded $B \subset \mathcal{F}$ and set $\mathcal{K} = B \cup \mathcal{K}_*$. If $T_{\omega_i} EU/a_i$ is bounded, Banach–Steinhaus on $\mathcal{H}$ gives an m for which $\sup_i \|T_{\omega_i} EU/a_i\|_{-m,\mathcal{K}} < \infty$. Equation (46) gives convergence uniformly on each bounded Schwartz test set and on B. Since B was arbitrary, this proves (47). □

Consequently, if the normalized net $T_{\omega_i} EU/a_i$ is bounded and converges to a nonzero distribution U^*, then, along the selected subnet, $T_{\omega_i} \mathcal{L}_{S,C,D_{S,C}} U/a_i$ converges to U^* as well.

7 Leading distributions and normalization scales

Fix $U \in \mathcal{A}$ and choose chambers $D_{S,C}$ as in Theorem 6.15. Fix $S \in \mathcal{S}_0 \setminus \{0\}$ and a full-dimensional cone C of $\mathcal{Q}|_S$. Write $Y = p_S\omega$ and $b = b_S\omega$, and put

$$F_{S,C} = \{\alpha \in \Gamma_{S,C,D_{S,C}} : \mathcal{C}_\alpha^{S,C} EU \neq 0\}.$$

For $\alpha \in F_{S,C}$, Lemma 6.5 gives a polynomial map

$$P_{S,C,\alpha} : S \longrightarrow \mathscr{P} \cap E_\alpha^S, \qquad P_{S,C,\alpha}(Y) := e^{\alpha(Y)} T_Y \mathcal{C}_\alpha^{S,C} EU.$$

The definition of $\mathcal{L}_{S,C,D_{S,C}}$ then gives

$$T_\omega \mathcal{L}_{S,C,D_{S,C}} U = T_b q_{S,C}(Y), \qquad q_{S,C}(Y) = \sum_{\alpha \in F_{S,C}} e^{-\alpha(Y)} P_{S,C,\alpha}(Y). \tag{48}$$

Here $q_{S,C} : S \to \mathscr{P}$. Let $E_{S,C,\alpha}$ be the real span of the coefficient distributions of $P_{S,C,\alpha}$, using the underlying real space for complex distributions. The coefficients of a vector-valued polynomial span the same space as its values. By the defining identity for $P_{S,C,\alpha}$, this is the real span of the S-dilates of $\mathcal{C}_\alpha^{S,C} EU$. Thus Lemma 6.3 gives

$$E_{S,C} = \bigoplus_{\alpha \in F_{S,C}} E_{S,C,\alpha} \subset \mathcal{S}'_Q(\mathcal{F}'_\beta). \tag{49}$$

Thus coefficient distributions belonging to distinct characters cannot satisfy a nontrivial linear relation. There may still exist nontrivial relations among coefficients of the same polynomial. However, since $q_{S,C}(Y) = T_Y \mathcal{L}_{S,C,D_{S,C}} U$ and T_Y is invertible, $q_{S,C}$ is either identically zero or nowhere zero.

7.1 Scales and limits of the finite residue formulas

Fix a norm on each $E_{S,C,\alpha}$ and give the direct sum $E_{S,C}$ the maximum norm

$$\left\| \sum_{\alpha \in F_{S,C}} u_\alpha \right\|_{S,C} := \max_{\alpha \in F_{S,C}} \|u_\alpha\|, \qquad u_\alpha \in E_{S,C,\alpha}.$$

The maximum over the empty set is interpreted as zero. Define

$$\boxed{M_{S,C}(Y) = \max_{\alpha \in F_{S,C}} e^{-\alpha(Y)} \|P_{S,C,\alpha}(Y)\| = \|q_{S,C}(Y)\|_{S,C}.} \tag{50}$$

If $F_{S,C}$ is nonempty, then $M_{S,C}(Y) > 0$ for every Y.

Theorem 7.1 (Uniform finite residue approximation)**.** Let $(\omega_i)_{i \in I}$ be a net in W. For each $i \in I$, let $S_i \in \mathcal{S}_0 \setminus \{0\}$ and put

$$Y_i = p_{S_i}\omega_i, \qquad b_i = b_{S_i}\omega_i.$$

Let C_i be a full-dimensional cone of $\mathcal{Q}|_{S_i}$ containing Y_i. Suppose (b_i) is bounded and $\mu_{S_i}(\omega_i) \to \infty$. If $a_i > 0$ and $T_{\omega_i} EU/a_i$ is bounded in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$, then

$$T_{\omega_i} EU = \sum_{\alpha \in F_{S_i,C_i}} e^{-\alpha(Y_i)} T_{b_i} P_{S_i,C_i,\alpha}(Y_i) + o_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(a_i).$$

Proof. Fix a bounded auxiliary test set $B \subset \mathcal{F}$, and enlarge it to $\mathcal{K}$ by the union of the finitely many detecting sets in Theorem 6.15, one for each (S, C). Banach–Steinhaus on $\mathcal{H}$ gives an order m such that $\sup_i \|T_{\omega_i} EU/a_i\|_{-m,\mathcal{K}} < \infty$. Increase m to meet the hypotheses of Theorem 6.15 for every pair (S, C). There are finitely many such pairs, so (46) gives common constants $M, A, \delta, R > 0$ and an output order $k \geq m$ with

$$\frac{\|T_{\omega_i} EU - T_{b_i} q_{S_i,C_i}(Y_i)\|_{-k,\mathcal{K}}}{a_i} \leq M e^{A\|b_i\| - \delta \mu_{S_i}(\omega_i)} \frac{\|T_{\omega_i} EU\|_{-m,\mathcal{K}}}{a_i}$$

whenever $\mu_{S_i}(\omega_i) \geq c\|b_i\| + R$. This condition holds eventually, and the right-hand side tends to zero. The estimate is uniform on each bounded Schwartz test set and on B. Since B was arbitrary, substituting (48) proves the assertion. □

Corollary 7.2 (Residue profiles along subnets)**.** Let $(\omega_i)_{i \in I}$ be a net in W with $\|\omega_i\| \to \infty$, and let $a_i > 0$. Suppose that $\{T_{\omega_i} EU/a_i : i \in I\}$ is bounded in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$ and that

$$T_{\omega_i} EU = a_i Z + o_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(a_i), \qquad Z \neq 0.$$

After passing to a subnet, there are $S \in \mathcal{S}_0 \setminus \{0\}$ and a full-dimensional cone C of $\mathcal{Q}|_S$ such that

$$b_i = b_S\omega_i \longrightarrow b, \qquad \mu_S(\omega_i) \longrightarrow \infty, \qquad Y_i = p_S\omega_i \in C.$$

There are $w_\alpha \in E_{S,C,\alpha}$, not all zero, such that

$$\frac{e^{-\alpha(Y_i)} P_{S,C,\alpha}(Y_i)}{a_i} \longrightarrow w_\alpha, \qquad Z = T_b \sum_{\alpha \in F_{S,C}} w_\alpha. \tag{51}$$

Moreover,

$$\frac{M_{S,C}(Y_i)}{a_i} \longrightarrow \max_{\alpha \in F_{S,C}} \|w_\alpha\| > 0. \tag{52}$$

For each α with $w_\alpha \neq 0$, there is a chamber $\mathfrak{c}_\alpha$ of the polar arrangement $\mathcal{P}$ such that

$$\alpha(Y) = \max_{z \in \overline{\mathfrak{c}_\alpha}} Y(z), \qquad Y \in C.$$

For each $\alpha \in F_{S,C}$, there is a nonempty flat L of $\mathcal{P}$ such that

$$Y|_L = \alpha(Y) \quad (Y \in S), \qquad \deg P_{S,C,\alpha} \leq m_U(L) - \operatorname{codim} L, \tag{53}$$

where m_U is defined in (25).

Proof. Lemma 6.1 gives a subnet and S with $b_i \to b$ and $\mu_S(\omega_i) \to \infty$. Restrict to a tail so that (b_i) is bounded. Then $\mu_S(Y_i) \to \infty$, so Y_i eventually avoids the cone walls. Pass to a further subnet on which Y_i lies in one cone C. Theorem 7.1 gives

$$T_{\omega_i} EU = T_{b_i} q_{S,C}(Y_i) + o_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(a_i).$$

Combining this with the hypothesis and applying T_{-b_i} yields

$$q_{S,C}(Y_i) = a_i T_{-b} Z + o_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(a_i).$$

Thus $q_{S,C}$ is nonzero. The space $E_{S,C}$ is closed and finite dimensional, so $T_{-b}Z \in E_{S,C}$. The projections onto the summands in (49) are continuous. Applying them gives (51), with at least one $w_\alpha \neq 0$. Since $F_{S,C}$ is finite and $a_i > 0$,

$$\frac{M_{S,C}(Y_i)}{a_i} = \max_{\alpha \in F_{S,C}} \left\| \frac{e^{-\alpha(Y_i)} P_{S,C,\alpha}(Y_i)}{a_i} \right\| \longrightarrow \max_{\alpha \in F_{S,C}} \|w_\alpha\| > 0.$$

Proposition 7.3 supplies the chamber for each $\alpha \in F_{S,C} \subseteq \Gamma_{S,C}$. For the degree bound, choose a threshold expansion of EU defining $\mathcal{C}_\alpha^{S,C} EU$ as in Lemma 6.5. Every terminal face F in its α-group has $S \subseteq W_F$ and $\ell_F|_S = \alpha$. Thus $P_{S,C,\alpha}$ is the sum of the corresponding $P_F U$ restricted to S. Apply (26) to each term. Neither restriction nor summation increases the maximum degree, so choose a face F attaining the largest bound among these finitely many terms and set $L = \operatorname{aff}(F)$. This proves (53). □

7.2 Linear programs for the characters

Proposition 7.3. For every $\alpha \in \Gamma_{S,C}$ there is a chamber $\mathfrak{c}_\alpha$ of $\mathcal{P}$ such that

$$\alpha(Y) = \max_{z \in \overline{\mathfrak{c}_\alpha}} Y(z) = \lambda_{\mathfrak{c}_\alpha}(Y), \qquad Y \in C. \tag{54}$$

Proof. Choose $U_\alpha \in \mathcal{A}$ with $\mathcal{C}_\alpha^{S,C} EU_\alpha \neq 0$, an objective $\theta \in C$, and a threshold above $\alpha(\theta)$. In the expansion of EU_α from Theorem 3.1, Lemma 6.5 identifies the nonzero α-group, which contains a residue term on an optimal face G of a processed chamber $\mathfrak{c}_\alpha$, with $\alpha(Y) = Y|_G$. Optimality puts θ in the cone generated by the outward facet normals at G. Choose independent such normals a_i such that θ is a positive combination of the a_i. Extend the a_i to a basis chosen from polar normal directions. The additional coordinates vanish at θ, hence on C by Lemma A.1 applied to their

coordinate hyperplanes. The selected coordinates cannot vanish anywhere on C: the same lemma would force them to vanish at θ. They therefore remain positive by connectedness of C. Writing the facet inequalities as $a_i(z) \le b_i$, we obtain $Y = \sum_i t_i(Y) a_i$ with $t_i(Y) > 0$ on C. Consequently

$$Y(z) \le \sum_i t_i(Y) b_i = Y|_G = \alpha(Y) \quad (z \in \overline{\mathfrak{c}_\alpha}),$$

with equality on G. This proves (54). □

7.3 Examples of scales and profiles

In these examples, take $S = W = \mathbb{R}^2$ and $C = (-\infty, 0)^2$. Write $\omega = (-t, -s)$ with $t, s > 0$, so the observation scales are (e^{-t}, e^{-s}) and $\omega(z, w) = -tz - sw$. Thus $Y_i = \omega_i$ and $b_i = 0$. For a Schwartz test h, the Mellin integrand is

$$e^{tz+sw} M_U(z, w) H_h(z, w), \qquad H_h(z, w) = \int_{(0,\infty)^2} x^{-z-1} y^{-w-1} h(x, y)\, dx\, dy.$$

We compute its residues in the negative tube, where H_h is holomorphic, and identify the resulting functionals of h.

Example 7.4 (Two comparable residues)**.** Let $U = (x + y)\mathbf{1}_{(0,1)^2}\, dx\, dy$. Its Mellin transform is

$$M_U(z, w) = \frac{1}{(z + 2)(w + 1)} + \frac{1}{(z + 1)(w + 2)}.$$

Start in the chamber $\mathfrak{c}_0 = \{-1 < z < 0,\ -1 < w < 0\}$. The polar lines of M_U are $z = -2, -1$ and $w = -2, -1$. The maximum of $-tz - sw$ on $\overline{\mathfrak{c}_0}$ is attained at $(-1, -1)$, but its double residue is zero: neither summand has a pole in both variables there. The two adjacent chambers across $z = -1$ and $w = -1$ are

$$\mathfrak{c}_1 = \{-2 < z < -1,\ -1 < w < 0\}, \qquad \mathfrak{c}_2 = \{-1 < z < 0,\ -2 < w < -1\}.$$

For $t, s > 0$, the linear programs in Proposition 7.3 give

$$\max_{(z,w)\in\overline{\mathfrak{c}_1}} (-tz - sw) = 2t + s, \qquad \max_{(z,w)\in\overline{\mathfrak{c}_2}} (-tz - sw) = t + 2s,$$

attained at $(-2, -1)$ and $(-1, -2)$, respectively. Thus the characters are

$$\alpha_1(\omega) = -2\omega_1 - \omega_2, \qquad \alpha_2(\omega) = -\omega_1 - 2\omega_2.$$

Both poles are simple in each variable, so the corresponding double residues of the integrand are

$$e^{-2t-s} H_h(-2, -1), \qquad e^{-t-2s} H_h(-1, -2).$$

Since $H_h(-2, -1) = \langle x\, dx\, dy, h\rangle$ and $H_h(-1, -2) = \langle y\, dx\, dy, h\rangle$, this computes

$$P_{S,C,\alpha_1}(-t, -s) = x\, dx\, dy, \qquad P_{S,C,\alpha_2}(-t, -s) = y\, dx\, dy.$$

Write

$$V_1 = x\, dx\, dy, \qquad V_2 = y\, dx\, dy.$$

Thus

$$q_{S,C}(-t,-s) = e^{-2t-s}V_1 + e^{-t-2s}V_2.$$

The scale is

$$M_{S,C}(-t,-s) = \max(e^{-2t-s}\|V_1\|, e^{-t-2s}\|V_2\|).$$

For $\omega_i = (-t_i, -s_i)$ with $t_i, s_i \to \infty$, we obtain

$$T_{\omega_i} EU = e^{-\alpha_1(\omega_i)}V_1 + e^{-\alpha_2(\omega_i)}V_2 + o_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}\big(\max(e^{-\alpha_1(\omega_i)}, e^{-\alpha_2(\omega_i)})\big).$$

Suppose $t_i - s_i \to c \in \mathbb{R}$ and choose $a_i = e^{-\alpha_2(\omega_i)}$. Then

$$\frac{e^{-\alpha_1(\omega_i)}}{a_i} \longrightarrow e^{-c}, \qquad \frac{e^{-\alpha_2(\omega_i)}}{a_i} = 1,$$

so

$$T_{\omega_i} EU = a_i Z + o_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(a_i), \qquad Z = e^{-c}V_1 + V_2.$$

In Corollary 7.2, the block limits are $w_{\alpha_1} = e^{-c}V_1$ and $w_{\alpha_2} = V_2$, so

$$\frac{M_{S,C}(Y_i)}{a_i} \longrightarrow \max(e^{-c}\|V_1\|, \|V_2\|).$$

If instead $t_i - s_i \to +\infty$, the same normalization gives $w_{\alpha_1} = 0$, $w_{\alpha_2} = V_2$, and $Z = V_2$. If $t_i - s_i \to -\infty$, choose $a_i = e^{-\alpha_1(\omega_i)}$; then $w_{\alpha_1} = V_1$, $w_{\alpha_2} = 0$, and $Z = V_1$. Thus the relative growth of t_i and s_i determines which characters contribute to Z and which exponential supplies a_i.

Example 7.5 (Polynomial cancellation within one character)**.** Consider the signed measure

$$U = ((\log x)^2 + \log y)\mathbf{1}_{(0,1)^2}\, dx\, dy, \qquad M_U(z,w) = \frac{2}{(z+1)^3(w+1)} - \frac{1}{(z+1)(w+1)^2}.$$

The polar lines $z = -1$ and $w = -1$ meet at the single point $(-1,-1)$. For the initial chamber $\mathfrak{c}_0 = \{-1 < z < 0,\ -1 < w < 0\}$, the corresponding linear program is

$$\max_{(z,w)\in\overline{\mathfrak{c}_0}}(-tz - sw) = t + s \qquad (t, s > 0),$$

with maximizer $(-1,-1)$. Thus the character is $\alpha(\omega) = -\omega_1 - \omega_2$, so $\alpha(-t,-s) = t+s$. To compute its polynomial, put $f(z,w) = e^{tz+sw}H_h(z,w)$. The double residue is

$$(\partial_z^2 - \partial_w)f\Big|_{(-1,-1)} = e^{-t-s}\left[(\partial_z^2 - \partial_w)H_h + 2t\partial_z H_h + (t^2 - s)H_h\right]_{(-1,-1)}.$$

Differentiating the test transform shows that H_h, $\partial_z H_h$, and $(\partial_z^2 - \partial_w)H_h$ at $(-1,-1)$ pair h with $dx\,dy$, $-\log x\, dx\, dy$, and $((\log x)^2 + \log y)\, dx\, dy$, respectively. Hence

$$P_{S,C,\alpha}(-t,-s) = \big[(\log x)^2 + \log y - 2t\log x + t^2 - s\big]dx\, dy.$$

Write

$$V_1 = ((\log x)^2 + \log y)\, dx\, dy, \qquad V_2 = \log x\, dx\, dy, \qquad V_3 = dx\, dy.$$

Thus $q_{S,C}(-t,-s) = e^{-t-s}[V_1 - 2tV_2 + (t^2 - s)V_3]$. There is one character, so

$$M_{S,C}(-t,-s) = e^{-t-s}\|V_1 - 2tV_2 + (t^2 - s)V_3\|.$$

For $\omega_i = (-t_i, -s_i)$ with $t_i, s_i \to \infty$,

$$T_{\omega_i} EU = e^{-t_i - s_i}\left[V_1 - 2t_i V_2 + (t_i^2 - s_i)V_3\right] + R_i. \tag{55}$$

For every bounded test set $\mathcal{B}$ and every N,

$$\|R_i\|_{\mathcal{B}} \le C_{\mathcal{B},N} e^{-t_i - s_i}(1 + t_i + |t_i^2 - s_i|)(e^{-Nt_i} + e^{-Ns_i}).$$

Suppose $(t_i^2 - s_i)/t_i \to c \in \mathbb{R}$ and choose $a_i = t_i e^{-t_i - s_i}$. Then $P_{S,C,\alpha}(Y_i)/t_i \to -2V_2 + cV_3$, and the remainder is $o_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(a_i)$. Consequently

$$T_{\omega_i} EU = a_i Z + o_{\mathcal{S}'_Q(\mathcal{F}'_\beta)}(a_i), \qquad Z = -2V_2 + cV_3.$$

Here $w_\alpha = Z$, and $M_{S,C}(Y_i)/a_i \to \| - 2V_2 + cV_3\|$. If instead $|t_i^2 - s_i|/t_i \to \infty$ and $\operatorname{sgn}(t_i^2 - s_i) = \epsilon$ is fixed, choose $a_i = |t_i^2 - s_i| e^{-t_i - s_i}$. Then $w_\alpha = Z = \epsilon V_3$ and $M_{S,C}(Y_i)/a_i \to \|V_3\|$. The character α is the same in both regimes; the polynomial block determines the change in normalization and profile.

8 Polynomial bounds and leading homogeneity

The character polynomials of Section 7 are orbits of a finite-dimensional dilation action. We use this action to bound their norms above and below by powers of the distance to the subspace of translations that leave them unchanged, and to prove that every nonzero leading distribution is homogeneous in the limiting direction of the logarithmic scales.

8.1 The finite-dimensional dilation action

Fix S, C and $\alpha \in F_{S,C}$, and abbreviate

$$v = \mathcal{C}_\alpha^{S,C} EU, \qquad P(Y) = P_{S,C,\alpha}(Y), \qquad \mathcal{V} = E_{S,C,\alpha}, \qquad D = \deg P.$$

Thus $v \neq 0$, $P(0) = v$, and $\mathcal{V}$ is the real span of the values of P. The identity

$$P(Y + X) = e^{\alpha(X)} T_X P(Y) \tag{56}$$

makes $\mathcal{V}$ invariant under S-dilations. Set

$$R(X) = e^{\alpha(X)} T_X|_{\mathcal{V}}, \qquad N(X) = \left.\frac{d}{dt}\right|_{t=0} R(tX).$$

Use the coefficient-space norms fixed in Section 7 and restrict a fixed Euclidean norm on W to S. Operator norms are induced by the norm on $\mathcal{V}$.

Lemma 8.1. The map $N : S \to \operatorname{End}_{\mathbb{R}}(\mathcal{V})$ is linear, its values commute, and

$$R(Y) = e^{N(Y)}, \qquad P(Y) = e^{N(Y)} v, \qquad N(Y)^{D+1} = 0 \quad (Y \in S). \tag{57}$$

There is a constant $C_0 > 0$ such that

$$1 \le \|R(Y)\|_{\mathrm{op}} \le C_0 \|P(Y)\| \quad (Y \in S). \tag{58}$$

Proof. Choose a basis $w_j = P(X_j)$ of $\mathcal{V}$. Since $R(Y)w_j = P(Y + X_j)$, every matrix entry of $R(Y)$ is a polynomial of degree at most D. The dilation identity gives $R(Y + X) = R(Y)R(X)$ and $R(0) = I$. Differentiating this identity shows that N is linear, its values commute, and $\frac{d}{dt}R(tY) = N(Y)R(tY)$. Hence $R(tY) = e^{tN(Y)}$. Its polynomial degree is at most D, so differentiation $D + 1$ times at zero gives $N(Y)^{D+1} = 0$ as an operator on all of $\mathcal{V}$.

Commutation also gives

$$R(Y)w_j = R(X_j)P(Y).$$

The finitely many fixed operators $R(X_j)$ and the chosen basis therefore give a constant $C_0 > 0$ such that $\|R(Y)\|_{\mathrm{op}} \le C_0\|P(Y)\|$. Every eigenvalue of $R(Y)$ is one, so $\|R(Y)\|_{\mathrm{op}} \ge 1$, proving (58). □

8.2 Sharp degree bounds for the coefficient norm

Directions in $\ker N$ do not change P. To measure only parameters that affect the polynomial, put

$$K = \ker N, \qquad \rho(Y) = \operatorname{dist}(Y, K).$$

Proposition 8.2 (Two-sided polynomial bounds)**.** If $D \ge 1$, there are constants $c, C > 0$ such that

$$\boxed{c(1 + \rho(Y))^{1/D} \le \|P(Y)\| \le C(1 + \rho(Y))^{D} \qquad (Y \in S).} \tag{59}$$

Proof. The induced linear map $S/K \to N(S)$ is an isomorphism, so $\|N(Y)\|_{\mathrm{op}} \asymp \rho(Y)$. The finite exponential in (57) immediately gives the upper bound.

For the lower bound, the finite logarithm recovers the generator:

$$N(Y) = \sum_{k=1}^{D} \frac{(-1)^{k+1}}{k}(R(Y) - I)^k. \tag{60}$$

Indeed, $(R(Y) - I)^{D+1} = 0$, and the formal identities for exponential and logarithm hold modulo terms of degree $D + 1$. Since $\|R(Y)\|_{\mathrm{op}} \ge 1$, this gives

$$\rho(Y) \le C_1\|N(Y)\|_{\mathrm{op}} \le C_2\|R(Y)\|_{\mathrm{op}}^{D} \le C_3\|P(Y)\|^{D}.$$

Combining this with the constant lower bound in (58) proves (59). □

8.3 Homogeneity of limiting polynomial profiles

Lemma 8.3. Suppose $Y_i \in S$, $c_i > 0$, and

$$\|Y_i\| \longrightarrow \infty, \qquad \frac{Y_i}{\|Y_i\|} \longrightarrow \theta, \qquad \frac{P(Y_i)}{c_i} \longrightarrow w \neq 0.$$

Then $N(\theta)w = 0$. Equivalently,

$$T_{h\theta}w = e^{-h\alpha(\theta)}w \qquad (h \in \mathbb{R}). \tag{61}$$

Proof. Use an auxiliary Euclidean norm on $\mathcal{V}$ in this proof. Put $\widehat{w} = w/\|w\|$. The hypotheses imply that $(0, \theta, \widehat{w})$ belongs to the closure of the semialgebraic set

$$\mathcal{G} = \left\{ \left(\frac{1}{\|Y\|}, \frac{Y}{\|Y\|}, \frac{P(Y)}{\|P(Y)\|} \right) : Y \in S \setminus \{0\} \right\}.$$

The denominator $\|P(Y)\|$ never vanishes by Lemma 8.1. Analytic curve selection gives an arc $(r(t), \eta(t), z(t))$, analytic at $t = 0$, whose value there is $(0, \theta, \widehat{w})$ and which lies in $\mathcal{G}$ for $0 < t < \varepsilon$. In particular $r(t) > 0$ there. For some integer $q \geq 1$ and $a > 0$,

$$r(t) = at^q + O(t^{q+1}), \qquad Y(t) := \frac{\eta(t)}{r(t)} = a^{-1}t^{-q}\big(\theta + O(t)\big).$$

These functions have convergent Laurent expansions. Since P is polynomial, so does $P(Y(t))$. Its uniform positive lower bound and its normalized limit $\widehat{w}$ imply

$$P(Y(t)) = t^{-p}\big(w_0 + O(t)\big), \qquad p \geq 0, \quad w_0 = b\widehat{w}, \quad b > 0,$$

where p is an integer. Differentiating (56) in X at zero and applying the chain rule gives

$$\frac{d}{dt}P(Y(t)) = N(Y'(t))P(Y(t)).$$

The left side is $O(t^{-p-1})$. On the right side the Laurent expansions give

$$N(Y'(t))P(Y(t)) = -qa^{-1}t^{-p-q-1}N(\theta)w_0 + O(t^{-p-q}).$$

Multiplying the identity by t^{p+q+1} and taking $t \downarrow 0$ therefore yields $N(\theta)w_0 = 0$, hence $N(\theta)w = 0$. By linearity, $N(h\theta) = hN(\theta)$. Since $N(\theta)w = 0$,

$$e^{hN(\theta)}w = w + \sum_{k=1}^{D} \frac{h^k}{k!}N(\theta)^k w = w \qquad (h \in \mathbb{R}).$$

The definition of R and (57) therefore give

$$T_{h\theta}w = e^{-h\alpha(\theta)}R(h\theta)w = e^{-h\alpha(\theta)}e^{hN(\theta)}w = e^{-h\alpha(\theta)}w,$$

which is (61). □

8.4 Homogeneity of the leading distribution

Theorem 8.4 (Leading homogeneity)**.** Let $U \in \mathcal{A}$ and suppose $a_i > 0$ and

$$\|\omega_i\| \longrightarrow \infty, \qquad \frac{\omega_i}{\|\omega_i\|} \longrightarrow \theta, \qquad \frac{T_{\omega_i}EU}{a_i} \longrightarrow Z \neq 0 \quad \text{in } \mathcal{S}'_Q(\mathcal{F}'_\beta),$$

with the normalized family bounded in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$. There is a unique $\lambda \in \mathbb{R}$ such that

$$\boxed{T_{h\theta}Z = e^{-h\lambda}Z \quad (h \in \mathbb{R}), \qquad \frac{\log a_i}{\|\omega_i\|} \longrightarrow -\lambda.} \tag{62}$$

On any subnet selected as in Corollary 7.2, with fixed S, C and $b_S\omega_i \to b$, one has

$$\theta \in S, \qquad \lambda = \min_{\alpha \in F_{S,C}} \alpha(\theta). \tag{63}$$

Proof. First pass to a subnet as in Corollary 7.2, and abbreviate $F = F_{S,C}$, $P_\alpha = P_{S,C,\alpha}$, and $Y_i = p_S\omega_i$. Since $b_i = \omega_i - Y_i \to b$,

$$\frac{\|Y_i\|}{\|\omega_i\|} \longrightarrow 1, \qquad \frac{Y_i}{\|Y_i\|} \longrightarrow \theta \in S.$$

Corollary 7.2 gives

$$\frac{e^{-\alpha(Y_i)}P_\alpha(Y_i)}{a_i} \longrightarrow w_\alpha, \qquad T_{-b}Z = \sum_{\alpha\in F} w_\alpha \neq 0. \tag{64}$$

Thus F is nonempty. Abbreviate $M = M_{S,C}$ from (50) and put

$$m(Y) = \min_{\alpha\in F} \alpha(Y).$$

Lemma 8.1 and the polynomial upper bounds give constants $c, C > 0$ and an integer $d_0 \geq 0$ such that

$$ce^{-m(Y)} \leq M(Y) \leq C(1 + \|Y\|)^{d_0}e^{-m(Y)}.$$

Equation (52) gives $M(Y_i)/a_i \to \max_\alpha \|w_\alpha\| > 0$. Consequently

$$\log a_i = -m(Y_i) + O(\log(2 + \|Y_i\|)). \tag{65}$$

Dividing by $\|\omega_i\|$ proves the asserted scalar limit on this subnet, with $\lambda = \min_{\alpha\in F} \alpha(\theta)$.

If $\alpha(\theta) > \lambda$, then for some $\delta > 0$ and all sufficiently large i, $\alpha(Y_i) - m(Y_i) \geq \delta\|Y_i\|$. It follows that

$$\frac{e^{-\alpha(Y_i)}\|P_\alpha(Y_i)\|}{a_i} \leq C'(1 + \|Y_i\|)^{d_0}e^{-\delta\|Y_i\|} \longrightarrow 0,$$

so $w_\alpha = 0$. For every surviving block, set $c_i = a_i e^{\alpha(Y_i)}$ in Lemma 8.3. It gives

$$T_{h\theta}w_\alpha = e^{-h\alpha(\theta)}w_\alpha = e^{-h\lambda}w_\alpha.$$

Summing and commuting $T_{h\theta}$ with T_b proves the homogeneity of Z.

Since $Z \neq 0$, the real number λ is uniquely determined by this identity. Every subnet of the original net admits a further subnet selected by Corollary 7.2. The preceding argument applies there and must give the same λ. The scalar limit in (62) therefore holds on the original net as well. □

For $r > 0$, the anisotropic dilation $\delta_{r,\theta}(x) = (r^{\theta_1}x_1, \ldots, r^{\theta_d}x_d)$ equals $D_{-(\log r)\theta}$. Thus (62) can also be written

$$(\delta_{r,\theta})_*Z = r^\lambda Z \qquad (r > 0).$$

This is homogeneity in the limiting direction, which may lie on a cone wall. It need not hold separately in every coordinate whose logarithmic scale diverges. For example, along $\omega = (-t, -t^2)$ in Example 7.5, the limiting direction is $(0, -1)$ and the leading profile $-2\log x\, dx\, dy$ is homogeneous in that direction, although its x-dilations still contain a nonconstant polynomial factor.

A Auxiliary geometric and analytic arguments

A.1 Subspaces spanned by arrangement vectors

Lemma A.1. Let W be a finite-dimensional real vector space and let $\mathcal{A} \subset W$ be a finite spanning set. Let $\mathcal{Q}$ consist of all hyperplanes spanned by subsets of $\mathcal{A}$, and let C be a relatively open cone of the restriction of $\mathcal{Q}$ to a subspace of W. If M is spanned by a subset of $\mathcal{A}$, then M is an intersection of hyperplanes of $\mathcal{Q}$. If also $C \cap M \neq \varnothing$, then $\operatorname{span} C \subseteq M$.

Proof. Choose a basis $a_1, \dots, a_r$ of M from $\mathcal{A}$ and extend it to a basis $a_1, \dots, a_n$ of W from $\mathcal{A}$. Writing t_i for the corresponding coordinates gives

$$M = \bigcap_{i=r+1}^{n} \ker t_i.$$

Each $\ker t_i$ is spanned by the other basis vectors and belongs to $\mathcal{Q}$. A relatively open cone of the restricted arrangement that meets such a hyperplane lies in it. If $C \cap M \neq \varnothing$, then C therefore lies in every $\ker t_i$ with $i > r$, hence in M. Taking spans proves the assertion. □

A.2 Finite-dimensional factorization

Lemma A.2. Let V be finite dimensional and let $(\Lambda, \leq)$ be a partially ordered set in which every infinite sequence has an infinite nondecreasing subsequence. For linear maps $L_\alpha : V \to E_\alpha$, $\alpha \in \Lambda$, there is a finite $A \subseteq \Lambda$ such that

$$L_\beta = \sum_{\substack{\alpha \in A \\ \alpha \leq \beta}} B_{\beta\alpha} L_\alpha, \qquad B_{\beta\alpha} : \operatorname{im} L_\alpha \longrightarrow \operatorname{im} L_\beta.$$

Proof. We first choose a finite $A \subseteq \Lambda$ such that

$$\bigcap_{\substack{\alpha \in A \\ \alpha \leq \beta}} \ker L_\alpha \subseteq \ker L_\beta \qquad (\beta \in \Lambda), \tag{66}$$

where an empty intersection is V. If no such A exists, we can choose β_n successively so that

$$\bigcap_{\substack{i<n \\ \beta_i \leq \beta_n}} \ker L_{\beta_i} \not\subseteq \ker L_{\beta_n}.$$

Choose an infinite nondecreasing subsequence $\beta_{n_1} \leq \beta_{n_2} \leq \cdots$. Every earlier term of this subsequence occurs in the intersection used to choose β_{n_j}. Hence

$$K_j := \bigcap_{i=1}^{j} \ker L_{\beta_{n_i}} \subsetneq K_{j-1}, \qquad K_0 = V.$$

Each step strictly lowers the dimension, contradicting $\dim V < \infty$. This proves (66).

For each β, collect the selected maps into

$$F_\beta : V \longrightarrow \bigoplus_{\substack{\alpha \in A \\ \alpha \leq \beta}} \operatorname{im} L_\alpha, \qquad F_\beta(v) = (L_\alpha v)_\alpha.$$

By (66), $F_\beta(v) = F_\beta(w)$ implies $L_\beta v = L_\beta w$. Thus the linear map

$$G_\beta : \operatorname{im} F_\beta \longrightarrow \operatorname{im} L_\beta, \qquad G_\beta(F_\beta(v)) = L_\beta v$$

is well defined. Extend G_β to the displayed direct sum by setting it to zero on a complementary subspace, and let $B_{\beta\alpha}$ be its restriction to the α-summand. Then

$$L_\beta v = G_\beta(F_\beta(v)) = \sum_{\substack{\alpha\in A\\ \alpha\le\beta}} B_{\beta\alpha} L_\alpha v,$$

as required. □

B Analytic details for Mellin inversion

B.1 Holomorphic Mellin representatives

Lemma B.1. Suppose $w_c = e^{\langle c,y\rangle} v \in \mathcal{L}(\mathcal{S}(W), \mathcal{F}'_\beta)$ for every $c \in \mathfrak{c}_0$. Then the transforms in (13) have a unique holomorphic $\mathcal{F}'_\beta$-valued representative on $\mathfrak{c}_0 + iA$.

Proof. Fix $c_* \in \mathfrak{c}_0$ and choose $\varepsilon > 0$ such that $c_* + [-\varepsilon, \varepsilon]^d \subset \mathfrak{c}_0$. Put

$$T = \sum_{\sigma\in\{\pm1\}^d} w_{c_*+\varepsilon\sigma} \in \mathcal{L}(\mathcal{S}(W), \mathcal{F}'_\beta), \qquad k_z(y) = \frac{e^{\langle z-c_*, y\rangle}}{\prod_{j=1}^d 2\cosh(\varepsilon y_j)}.$$

Let $V = \{z \in A_{\mathbb{C}} : |\Re z_j - c_{*,j}| < \varepsilon \text{ for all } j\}$. On every compact subset of V, the functions k_z and their y-derivatives decay exponentially in y, uniformly in z. The same holds after multiplication by any polynomial in y. Taylor's formula therefore shows that $z \mapsto k_z$ is holomorphic as a map into $\mathcal{S}(W)$, with

$$\partial_z^\alpha k_z(y) = y^\alpha k_z(y).$$

Set $m(z) := T(k_z)$. For every bounded $B \subset \mathcal{F}$, continuity of T gives a continuous Schwartz seminorm p_B such that

$$\sup_{f\in B} |\langle T(g), f\rangle| \le p_B(g).$$

By linearity of T, for complex $t \to 0$,

$$\begin{aligned}
&\sup_{f\in B} \left| \left\langle \frac{m(z+te_j) - m(z)}{t} - T(y_j k_z), f \right\rangle \right| \\
&\qquad = \sup_{f\in B} \left| \left\langle T\left(\frac{k_{z+te_j} - k_z}{t} - y_j k_z \right), f \right\rangle \right| \\
&\qquad \le p_B\left(\frac{k_{z+te_j} - k_z}{t} - y_j k_z \right) \longrightarrow 0.
\end{aligned}$$

The last limit follows from differentiability of k_z in $\mathcal{S}(W)$. Since B was arbitrary, this proves $\partial_{z_j} m(z) = T(y_j k_z)$ in the topology of $\mathcal{F}'_\beta$. Iterating on each index gives $\partial_z^\alpha m(z) = T(y^\alpha k_z)$, which is continuous. Thus m is holomorphic.

For real $c \in V$, the definitions give $k_c T = w_c$ on compactly supported tests. Since both sides are tempered, the identity holds on all Schwartz tests. By the Leibniz rule, each Schwartz seminorm of $k_{c+i\xi}$ grows at most polynomially in ξ. Thus, for $\varphi \in \mathcal{S}(A)$, the integral

$$\int_A k_{c+i\xi}\varphi(\xi)\,d\xi = k_c\widehat{\varphi}(-\cdot)$$

converges in $\mathcal{S}(W)$. Applying T and pairing with $f \in \mathcal{F}$ yields

$$\begin{aligned}\int_A \langle m(c+i\xi), f\rangle\varphi(\xi)\,d\xi &= \big\langle T(k_c\widehat{\varphi}(-\cdot)), f\big\rangle \\ &= \big\langle w_c(\widehat{\varphi}(-\cdot)), f\big\rangle.\end{aligned}$$

This is the representation identity for (13). The local holomorphic representatives agree on overlaps by uniqueness of continuous representatives of distributions, and therefore define a unique holomorphic representative on $\mathfrak{c}_0 + iA$. □

B.2 The cutoff limit

Lemma B.2. For $c \in (-\infty, 0)^d \subset A$, the map

$$A_c : \mathcal{H} \longrightarrow \mathcal{S}(W), \qquad A_c h(y) = e^{-\langle c,y\rangle} h(e^y),$$

is continuous. More precisely, for every compact $K \subset (-\infty, 0)^d$ and every Schwartz seminorm q, there are finitely many seminorms $p_{N,\alpha}$ and a constant C such that

$$\sup_{c\in K} q(A_c h) \le C \sum_{N,\alpha} p_{N,\alpha}(h).$$

Proof. The chain rule gives

$$\partial_y^\alpha A_c h = e^{-\langle c,y\rangle}\left[\prod_i (x_i\partial_{x_i} - c_i)^{\alpha_i} h\right](e^y).$$

The bracket is a finite sum of polynomial multiples of derivatives of h, with coefficients bounded for $c \in K$. On each orthant in y, the weight decays exponentially in the negative coordinates, uniformly over K. In the positive coordinates, sufficiently many powers of $(1 + \|e^y\|)^{-1}$ dominate both the weight and the polynomial factors. These bounds also absorb any power of $1 + \|y\|$, proving the asserted seminorm estimate. □

Use the cutoffs of Section 4.2, with bounded Euler derivatives.

Proof of Proposition 4.1. Let $U \in \operatorname{dom} E$ and fix a cutoff χ. The definition of $\operatorname{dom} E$ gives $EU \in \mathcal{S}'_Q(\mathcal{F}'_\beta)$ and the asserted convergence in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$. The cutoffs are eventually one on compactly supported tests in X, so $(EU)|_X = U|_X$. Thus $EU = 0$ implies that U is supported on ∂Q. Conversely, every such distribution is annihilated by each χ_n, proving the kernel formula. For $U \in \operatorname{dom} E$, the difference $U - EU$ is supported on ∂Q. Hence $\chi_n EU = \chi_n U$ for every cutoff χ and every $n \ge 1$. Taking limits proves that $EU \in \operatorname{dom} E$ and $E(EU) = EU$.

□

Proof of Proposition 4.2. Use A_c from Lemma B.2. The cutoff error is

$$A_c(\chi_n h) - A_c h = (\chi_n(e^y) - 1)A_c h.$$

Since $1 - \chi_n(e^y)$ is supported where some $y_i \le -\log n + C$, we have $\chi_n(e^y) \to 1$, with equality to 1 on every fixed compact set for sufficiently large n. By the assumed bounds on the Euler derivatives of χ, all y-derivatives of $\chi_n(e^y)$ are bounded uniformly in n. On this escaping set, the exponential decay in the negative logarithmic coordinate absorbs every polynomial weight in y. The estimates in Lemma B.2, with higher Schwartz orders on h, therefore give

$$A_c(\chi_n h) \longrightarrow A_c h \quad \text{in } \mathcal{S}(W),$$

uniformly for h in each bounded subset of $\mathcal{H}$. The map $h \mapsto w_c(A_c h)$ is continuous into $\mathcal{F}'_\beta$ and supported in Q. For every bounded $B \subset \mathcal{F}$ there is a Schwartz seminorm p_B with

$$\sup_{f \in B} |\langle w_c(g), f\rangle| \le p_B(g).$$

Thus the preceding convergence remains uniform after application of w_c and evaluation on B. Fix $n \ge 1$. Choose $\rho \in C_c^\infty(\mathbb{R}^d)$ equal to one near zero and put $\rho_R(x) = \rho(x/R)$. Since $\chi_n h$ vanishes near the coordinate boundary, $\rho_R \chi_n h$ is compactly supported in X. The definition $w_c = e^{\langle c,y\rangle}\ell_*(U|_X)$ therefore gives

$$\langle U, \rho_R \chi_n h\rangle = \langle w_c, A_c(\rho_R \chi_n h)\rangle.$$

As $R \to \infty$, Schwartz decay gives $\rho_R \chi_n h \to \chi_n h$ in $\mathcal{H}$, and continuity of A_c gives $A_c(\rho_R \chi_n h) \to A_c(\chi_n h)$ in $\mathcal{S}(W)$. Continuity of U and w_c then yields

$$\langle U, \chi_n h\rangle = \langle w_c, A_c(\chi_n h)\rangle.$$

Now let $n \to \infty$. The uniform bounds above show convergence in $\mathcal{S}'_Q(\mathcal{F}'_\beta)$ to the continuous operator $h \mapsto w_c(A_c h)$. This operator is independent of χ, proving $U \in \operatorname{dom} E$ and (18). □

B.3 Mellin transforms of test functions

Proof of Lemma 4.3. For an integer $R \ge 0$, put

$$P_R(z) = \prod_i \prod_{n=0}^{R-1} (z_i - n), \qquad L_R = \prod_i \prod_{n=0}^{R-1} (x_i \partial_{x_i} - n).$$

Since $\prod_{n=0}^{R-1}(x_i\partial_{x_i} - n) = x_i^R \partial_{x_i}^R$, the function $L_R h$ vanishes to order R at each coordinate face. Thus

$$F_R(z) := \int_X x^{-z}(L_R h)(x)\, d^\times x$$

is holomorphic on $\Re z_i < R$ for every i. Integration by parts on the initial tube $\Re z_i < 0$ gives $F_R = P_R H_h$, so F_R/P_R continues H_h meromorphically to the larger tube. Taking $R \to \infty$, H_h meromorphically continues to $A_{\mathbb{C}}$. Each factor $z_i - n$ occurs only once in P_R, so the pole order along each coordinate hyperplane is at most one.

For the decay bounds, fix a compact set $K \subset A$ and choose $R > \max_{c \in K,\, i} c_i$. Changing variables $x = e^y$ gives

$$F_R(c + i\xi) = \widehat{g_c}(\xi), \qquad g_c(y) = e^{-\langle c,y\rangle}(L_R h)(e^y).$$

Writing $\mathbf{1} = (1, \dots, 1)$, we have

$$g_c = A_{c - R\mathbf{1}}(\partial_{x_1}^R \cdots \partial_{x_d}^R h).$$

Since $K - R\mathbf{1}$ is a compact subset of $(-\infty, 0)^d$, Lemma B.2 bounds every Schwartz seminorm of g_c, uniformly over K, by finitely many seminorms of h. Continuity of Fourier transformation then gives, for every N,

$$\sup_{c \in K,\, \xi \in A} (1 + \|\xi\|)^N |F_R(c + i\xi)| \le C\, p(h),$$

where p is a Schwartz seminorm and C is independent of h. At distance at least $\delta > 0$ from the polar hyperplanes, each factor of P_R has modulus at least δ, so $|H_h| \le \delta^{-dR} |F_R|$. Thus every decay seminorm (2) of H_h is bounded by a constant times a Schwartz seminorm of h. This proves continuity of $h \mapsto H_h$ into $\Omega^0_{\mathrm{rd}}(A, \mathcal{P}_0)$; linearity follows from the defining integral. $\square$

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