Meromorphic Continuation of Archimedean Zeta Functions

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1 Setting and Main Theorem

Consider the following setting.

1. $\Omega \subset \mathbb{R}^d$ is open and $W \subset \Omega$ is compact.
2. $K : \Omega \to \mathbb{R}$ is analytic.
3. $\phi : \Omega \to \mathbb{C}$ is infinitely differentiable with $\operatorname{supp} \phi = W$.
4. Using a branch of the complex logarithm function that is continuous on $\mathbb{R} \setminus \{0\}$, we define $\zeta(z) = \int_W K(w)^z \phi(w) dw$.

Theorem 1.1 (Main Theorem)**.** *ζ has an analytic continuation to a meromorphic function on $\mathbb{C}$.*

The theorem was first conjectured by Gelfand[3] in 1954 in the case that K is a non-negative polynomial. Gelfand proposed the problem to investigate the notion of assigning sensible finite values to divergent integrals. Let's clarify this notion with an example.

Example 1.2. *Let $W = [0,1]$. Let $\phi = x^n$ and $K(w) = x^k$ for $n, k \in \mathbb{R}$ and $k \neq 0$. Then $\zeta(z) = \int_W K(w)^z \phi(w) dw$ has an analytic continuation to $\mathbb{C}$ given by $\frac{1}{kz+n+1}$.*

Proof. Explicitly compute the integral for $kz + n > 0$. $\int_{[0,1]} (x^k)^z x^n dx = \int_{[0,1]} x^{kz+n} dx = \frac{1}{kz+n+1}$. This formula defines a meromorphic function on $\mathbb{C}$ that is equal to $\zeta(z)$ on $kz + n > 0$. □

Now consider the formal integral $\int_{[0,1]} x^{-2} dx$. Of course, this diverges. But consider the function $\zeta(z) = \int_{[0,1]} x^z dx$. $\zeta(z)$ has an analytic continuation given by $\frac{1}{z+1}$. As formal integrals, $\int_{[0,1]} x^{-2} dx = \int_{[0,1]} x^z dx|_{z=-2}$. So we can sensibly evaluate $\frac{1}{z+1}|_{z=-2}$ and assign the value -1 to $\int_{[0,1]} x^{-2} dx$.

Atiyah[1] first proved Gelfand's conjecture in 1970 using resolution of singularities. Then Bernstein[2] proved it in 1972 using the Bernstein-Sato polynomial. Greenblatt[4], León-Cardenal[6], and Watanabe[7] have also considered similar settings. I present my own proof, primarily inspired by Watanabe.

2 Motivation from Probability and Statistics

If we consider w as a random vector with density function ϕ, evaluations of ζ gives the moments of $K(w)$, and thus ζ contains all the information about its distribution. In fact, ζ is the moment generating function of $\log(K(w))$. So ζ provides a lens to understand a random variable similar to its moment generating function.

For large z, ζ describes the growth rate of $K(w)$ towards infinity, and for negative z, ζ describes the decay rate of $K(w)$ towards 0. The Main Theorem tells us in particular that locally around the first negative singularity at $-\lambda$, $\zeta(z) = \frac{a}{(z+\lambda)^m} + O(1)$. This provides an excellent understanding of the decay rate of $K(w)$. Note that from inspection of the proof given below, every pole is a negative rational number, so we can speak of a first singularity.

In singular learning theory[7], we regard W as the space of parameters of a Bayesian model, $\phi(w)$ as the prior, and $K(w) = E_x[f(x, w)]$ as the KL divergence from the true distribution to the model. The asymptotics of a learning machine is dominated by neighborhoods of the zeros, just like the poles of ζ will be determined by neighborhoods of the zeros. $\zeta(z) = \frac{a}{(z+\lambda)^m} + O(1)$ is used to derive that $-\log \int_W e^{-nK(w)} \phi(w) dw = \lambda \log n - (m-1) \log \log n + O(1)$. Under some regularity assumptions on the distribution of f, this leads to a formula for the generalization error of the Bayesian posterior $\frac{\lambda}{n} + o(\frac{1}{n})$. We can consider λ, which we recall is the negative of the greatest singularity of ζ, to measure the complexity of the Bayesian model.

3 Preparation

For the problem to make sense, we first have to check that $\zeta(z)$ is holomorphic on some open set.

Proposition 3.1. *Let W be compact. Let $K : W \to \mathbb{R}$ be continuous. Let $\phi : W \to \mathbb{C}$ be continuous. Then $\zeta(z) = \int_W K(w)^z \phi(w) dw$ is holomorphic on the open right half plane.*

Proof. Because W is compact, K and ϕ are bounded. So for $\operatorname{Re}(z) > 0$, $K(w)^z \phi(w)$ is bounded. So the integral is finite.

Define $f(w, z) = K(w)^z \phi(w)$. For $K(w) = 0$, $\frac{\partial f}{\partial z} = 0$ and $K(w) \neq 0$, $\frac{\partial f}{\partial z} = \log(K(w)) K(w)^z \phi(w)$. This is continuous in z and w. Since W is compact, $\frac{\partial f}{\partial z}$ is bounded by some M in a neighborhood of $z > 0$. So $|\frac{f(w,z+h)-f(w,z)}{h}| \leq M$ for small h. Then by the bounded convergence theorem, $\lim_{h\to 0} \frac{\zeta(z+h)-\zeta(z)}{h} = \lim_{h\to 0} \int_W \frac{f(w,z+h)-f(w,z)}{h} dw = \int_W \lim_{h\to 0} \frac{f(w,z+h)-f(w,z)}{h} dw = \int_W \frac{\partial f}{\partial z} dw$. □

Corollary 3.2. *Let W be compact. Let $K : W \to \mathbb{R}$ be continuous and have no zeros. Let $\phi : W \to \mathbb{R}$ be continuous. Then $\zeta(z) = \int_W K(w)^z \phi(w) dw$ is an entire function.*

Proof. Since $K \neq 0$ and W is compact, K is bounded away from 0. Defining f as from the proof of Proposition 3.1, $\frac{\partial f}{\partial z} = \log(K(w))K(w)^z\phi(w)$ is bounded uniformly in a neighborhood of $z \in \mathbb{C}$. The rest of the proof is the same. □

In the following cases of monomials, we can prove the existence of a meromorphic continuation to $\mathbb{C}$ by integrating by parts.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^d$ be open. Let W be a rectangle contained in Ω. Let $\phi(w) = \psi(w) \prod_{i=1}^d |w_i|^{h_i}$ with $\psi : \Omega \to \mathbb{C}$ infinitely differentiable. Let $K(w) = \prod_{i=1}^d w_i^{k_i}$ for non-negative integers k_i and h_i. Then $\zeta(z) = \int_W K(w)^z \phi(w) dw$ meromorphically continues to $\mathbb{C}$.*

Proof. By choice of basis, we can assume that W is axis aligned. By separately considering finitely many subrectangles on which $w_i \geq 0$ or $w_i \leq 0$, we instead compute $\int_W \prod_{i=1}^d (w_i^{k_i})^z w_i^{h_i} \psi(w) dw$. Every rectangle can be counted once by adding and subtracting finitely many subrectangles with a corner at 0, so it is sufficient to assume there is a corner at 0. Rescaling the ith dimension by $c \neq 0$ pulls out a factor of $(c^{k_i})^z c^{h_i+1}$, which is holomorphic. Combining these observations, we can assume the rectangle is the unit box.

Then since $w_i \geq 0$, $(w_i^{k_i})^z w_i^{h_i} = w_i^{k_i z + h_i}$. So $\zeta(z) = \int_{[0,1]^d} \prod_{i=1}^d w_i^{k_i z + h_i} \psi(w) dw$. Using that ψ is infinitely differentiable, we integrate by parts on each dimension when $\operatorname{Re} z > 0$.

$$
\begin{aligned}
f(w) &= \prod_{i=1}^d w_i^{k_i z + h_i} \\
F(w) &= \prod_{i=1}^d \frac{w_i^{k_i z + h_i + 1}}{k_i z + h_i + 1} \\
g(w) &= \frac{\partial}{\partial w_i \dots \partial w_d} \psi(w) \\
G(w) &= \psi(w) \\
\zeta(z) &= \int_{[0,1]^d} f(w) G(w) dw \\
&= \sum_{\alpha \in \{0,1\}^d} (-1)^{|\alpha|} F(\alpha) G(\alpha) + (-1)^d \int_{[0,1]^d} F(w) g(w) dw \\
&= \sum_{\alpha \in \{0,1\}^d} (-1)^{|\alpha|} \prod_{i=1}^d \frac{\alpha_i^{k_i z + h_i + 1}}{k_i z + h_i + 1} \psi(\alpha) + (-1)^d \prod_{i=1}^d \frac{1}{k_i z + h_i + 1} \int_{[0,1]^d} \prod_{i=1}^d w_i^{k_i z + h_i + 1} g(w) dw
\end{aligned}
$$

The first term has a meromorphic continuation to $\mathbb{C}$. The second term has two factors. The first factor has a meromorphic continuation to $\mathbb{C}$. The second factor is of the same form as ζ. So we can integrate by parts n times (nd times including each dimension), pull out meromorphic factors, and be left to consider just a term of the form

$$\int_{[0,1]^d} \prod_{i=1}^d w_i^{k_i z + h_i + n} \theta(w) dw$$

Which is holomorphic on $\operatorname{Re} z > -\min_{1 \leq i \leq d} \frac{h_i + n}{k_i}$ by Proposition 3.1, assuming $k_i \neq 0$ for some i. So for all $n \in \mathbb{N}$, ζ has a meromorphic continuation to $\{z \in \mathbb{C} : \operatorname{Re} z > -\min_{1 \leq i \leq d} \frac{h_i + n}{k_i}\}$. Taking n towards infinity, ζ has a meromorphic continuation to $\mathbb{C}$. In the case that $k_i = 0$ for all i, ζ is constant, thus holomorphic. □

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^d$ be open and $W \subset \Omega$ be compact. Let $\phi(w) = \psi(w) \prod_{i=1}^d |w_i|^{h_i}$, where h_i are non-negative integers and $\psi : \Omega \to \mathbb{C}$ is infinitely differentiable with support W. Let $K(w) = \prod_{i=1}^d w_i^{k_i}$ for non-negative integers k_i. Then $\zeta(z) = \int_W K(w)^z \phi(w) dw$ meromorphically continues to $\mathbb{C}$.*

Proof. Since $\psi : \Omega \to \mathbb{C}$ is infinitely differentiable with compact support W, we can smoothly extend ψ to a rectangle R containing $\operatorname{supp} \psi$ while maintaining $\operatorname{supp} \psi = W$ by simply setting $\psi(w) = 0$ outside W. Because the support is unchanged, $\int_W K(w)^z \phi(w) dw = \int_R K(w)^z \phi(w) dw$. We conclude by Proposition 3.3.

Alternatively, the same strategy as the proof of Proposition 3.3 could be used. In this case, the boundary terms from integrating by parts disappear because ϕ vanishes on the boundary of its support. □

Since we can always split the integral into a part near the zeros and a part away from the zeros, Corollary 3.2 tells us that the singularities are determined only by the behavior around the zeros. Proposition 3.3 and Corollary 3.4 prove the theorem for important cases of monomials, which are functions that characterize power law decay rates. These observations suggest a proof strategy:

1. Identify the set of zeros.
2. Find the power law growth rate of K around the zeros.
3. Replace K with a monomial with the appropriate power law.

4 Univariate case

In the univariate case, we can regard K as a holomorphic function. We can use local properties of holomorphic functions to execute the proof strategy.

1. Holomorphic functions have finitely many zeros on a compact set.
2. The growth rate of a non-constant holomorphic function around a zero always follows a power law according to its first non-zero derivative.
3. Lemma 4.1 will allow us to perform a change of coordinates such that K is a monomial.

Lemma 4.1. *Let f be a holomorphic function on a domain Ω. Let c be an order k root of f. In a neighborhood of c, f has a kth root h. Locally, h is invertible. Additionally, if f is real analytic and c is real, we can choose h to be real analytic.*

Proof. We can assume $c = 0$. Locally, $f(w) = \alpha(w)w^k$, where α is holomorphic and nonzero. Since $\alpha \neq 0$ and we can choose a simply connected neighborhood, there is a holomorphic function v such that $e^{v(w)} = \alpha$. In particular, we can choose v as any primitive of $\frac{\alpha'}{\alpha}$. If α is real analytic, $\frac{\alpha'}{\alpha}$ is real on $\mathbb{R}$, so we can choose v to be real on $\mathbb{R}$. Then define $p = e^{\frac{1}{k}v(w)}$. $p^k = \alpha$ and p is real analytic if v is. Then $h = pw$ is a kth root of f and real analytic if p is. $h' = p'z + p$, so $h'(0) \neq 0$. By the inverse function theorem, h is locally invertible. □

Proposition 4.2. *Let the assumptions of 1.1 hold with $d = 1$. Then the conclusion of the 1.1 holds.*

Proof. Since K is analytic in one real variable, we can consider it a holomorphic function in $\mathbb{C}$. If K is constant, ζ is constant, thus holomorphic.

Assume K is non-constant. Since W is compact, K has finitely many zeros in W. Around each zero c_i, by Proposition 4.1, K has a holomorphic k_ith root h_i in a neighborhood O_i of c_i, where k_i is the order of the zero. There is a real interval $I_i \subset O_i$ containing c_i.

Then we split the integral $\int_W K(w)^z\phi(w)dw = \sum_{i=1}^N \int_{I_i} K(w)^z\phi(w)dw + \int_{W\setminus\bigcup_{i=1}^N I_i} K(w)^z\phi(w)dw$.

On $W \setminus \bigcup_{i=1}^N I_i$, $K(w) \neq 0$, so that term is holomorphic by Proposition 3.2. For the other terms, because h_i is invertible, we can perform a change of variables. Because h_i is real analytic, the resulting integral is over $\mathbb{R}$.

$$\begin{aligned}\int_{I_i} K(w)^z\phi(w)dw &= \int_{I_i} ((h_i(w))^{k_i})^z\phi(w)dw \\ &= \int_{h_i(I_i)} (u^{k_i})^z\phi(h_i^{-1}(u))|(h_i^{-1})'(u)|du\end{aligned}$$

We can take I_i small enough that $|(h_i^{-1})'(u)| \neq 0$, so $\phi(h_i^{-1}(u))|(h_i^{-1})'(u)|$ is infinitely differentiable. So this term analytically continues to a meromorphic function by Proposition 3.3. □

5 Multivariate Case

Two challenges arise in generalizing this strategy to multiple dimensions. First, the zeros of an analytic function of multiple variables are not necessarily a finite set, but are in general an analytic set, which can be much more complicated. Second, an analytic function of multiple variables may not have a growth rate that is well described by the same power law in all directions. For example, $K(x, y) = x^2y^4$ grows quadratically in the direction of x and quartically in the direction of y. These are questions of study in algebraic geometry. Hironaka's Resolution Theorem will simplify both challenges:

Theorem 5.1. *[Hironaka's Theorem[7]] Let $\Omega \subset \mathbb{R}^d$ be open and contain c. Let $K : \Omega \to \mathbb{R}$ be analytic, non-constant, and $K(c) = 0$.*

Then there exists a triple (O, M, g) where

1. *$O \subset \Omega$ is a neighborhood of c*
2. *M is a d-dimensional real analytic manifold*
3. *$g : M \to O$ is a real analytic map*

which satisfies the following conditions

1. *For any compact set $C \subset O$, $g^{-1}(C)$ is a compact set in M.*
2. *We use the notation $O^0 = \{x \in O : f(x) = 0\}$ and $M^0 = \{u \in M : K(g(u)) = 0\}$. g is a real analytic isomorphism of $O \setminus O^0$ and $M \setminus M_0$.*
3. *For an arbitrary point $p \in M^0$, there is a local coordinate $u = (u_1, u_2, \ldots, u_d)$ of M with coordinate map γ in which $\gamma(p) = 0$, $K(g(\gamma^{-1}(u))) = S\prod_{i=1}^d u_i^{k_i}$, where $S = \pm 1$ is constant and $k_1, k_2, \ldots, k_d$ are non-negative integers, and the Jacobian determinant of $g \circ \gamma^{-1}$ is $b(u)\prod_{i=1}^d |u_i|^{h_i}$, where $b(u) \neq 0$ is a real analytic function and $h_1, h_2, \ldots, h_d$ are non-negative integers.*

This is a vast generalization of Proposition 4.1. It will conceptually play the same role in our proof of the Main Theorem. In spirit, our proof follows the proof sketch from section 3.

1. Cover the zeros of f with finitely many open sets O on which we apply Hironaka's theorem.
2. Hironaka's theorem reveals the power law behavior in d directions around any zero.
3. By change of coordinates $O \to_{g^{-1}} M \to_{\text{chart}} \mathbb{R}^d$, Hironaka's theorem locally expresses K as a monomial. Then we can locally integrate around the zeros explicitly.

An issue with this plan is that the open sets on which we apply Hironaka's theorem are typically not disjoint, and the open sets on which the coordinate charts of M are defined are typically not disjoint. We need to avoid integrating the same region multiple times. Partitions of unity are a natural solution to this concern. For the readers unfamiliar,

Definition 5.2 (Partition of Unity)**.** *Let I be an index set and X a topological space. Let $\{\rho_i\}_{i\in I}$ be a set of continuous functions $\rho_i : X \to \mathbb{R}$. If for all $i \in I$, $0 \le \rho_i \le 1$, and for all $x \in X$, $\sum_{i\in I} \rho_i = 1$, $\{\rho_i\}_{i\in I}$ is a partition of unity.*

In a space with a smooth structure, if the functions are smooth, the partition of unity is called smooth.

Proposition 5.3. *[5] Let M be a smooth manifold and $\{U_i\}_{i\in I}$ an open cover of M. Then there exists a smooth partition of unity $\{\rho_i\}_{i\in I}$ such that* $\operatorname{supp}\rho_i \subset U_i$ *and for any $x \in X$, there is a neighborhood U_x such that only finitely many ρ_i are nonzero anywhere in U_x.*

We will not need the last property, but it is critical that $\operatorname{supp} f_i \subset U_i$ so that whenever $\operatorname{supp} f_i \neq 0$, we will be able to use the coordinate map defined on U_i.

Theorem 5.4 (Main Theorem)**.**

Proof. If K is constant, ζ is constant, thus holomorphic. Otherwise, around each root c of K, we can apply Hironaka's Theorem, obtaining O_c, M_c, g_c. Additionally, let γ_c be the coordinate chart described in Hironaka's theorem. Then $\gamma_c \circ g_c^{-1}$ is an analytic isomorphism $U_c \setminus U^0 \to V_c$, where U_c is a neighborhood of c and $V_c \subset \mathbb{R}^n$.

Because W is compact and $K^{-1}(0)$ is closed, $K^{-1}(0)$ is compact, so we can choose a finite subcover of $\{U_c\}_{K(c)=0}$. We label this subcover $\{U_i\}_{i=1}^N$.

$W \setminus \bigcup_{i=1}^N U_i$ is closed, thus compact, and contains no zeros. So there is nonzero minimum $\epsilon = \inf\{|K(w)| : w \in W \setminus \bigcup_{i=1}^N U_i\}$. Define $U_{N+1} = \{w \in W : K(w) > \epsilon/2\}$. Then $\{U_i\}_{i=1}^{N+1}$ is an open cover of W.

Now by Proposition 5.3, let $\{\rho_i\}_{i=1}^{N+1}$ be a smooth partition of unity for $\{U_i\}_{i=1}^{N+1}$ such that $\operatorname{supp}\rho_i \subset U_i$.

$$\begin{aligned}
\zeta(z) &= \int_W K(w)^z \phi(w) dw \\
&= \int_W K(w)^z \phi(w) \sum_{i=1}^{N+1} \rho_i(w) dw \\
&= \sum_{i=1}^{N+1} \int_W K(w)^z \phi(w) \rho_i(w) dw
\end{aligned}$$

So it is sufficient to show $\int_W K(w)^z \phi(w)\rho_i(w)dw$ has a meromorphic continuation to $\mathbb{C}$ for all i.

For $i = N+1$, $\int_W K(w)^z \phi(w)\rho_{N+1}(w)dw$ analytically continues to an entire function by Proposition 3.2.

Otherwise, i corresponds to a resolution of singularities. Since we now only need to work with a single i, from now on we drop the index from all notation.

$\operatorname{supp}\rho \subset U$ and $\gamma \circ g^{-1}$ is an analytic isomorphism $U \setminus U^0 \to V$. Since U^0 is a null set, we can ignore it in the integral. By the change of variables formula,

$$\begin{aligned}
\int_W K(w)^z \phi(w)\rho(w)dw &= \int_{U_i \setminus U^0} K(w)^z \phi(w)\rho(w)dw \\
&= \int_V K(g(\gamma^{-1}(u)))^z \phi(g(\gamma^{-1}(u)))\rho(g(\gamma^{-1}(u)))|\det J_{g\circ\gamma^{-1}}(u))|du \\
&= \int_V (S\prod_{j=1}^d u_j^{k_j})^z \phi(g(\gamma^{-1}(u)))\rho(g(\gamma^{-1}(u)))b(u)\prod_{j=1}^d |u_j|^{h_j} du \\
&= S^z \int_V (\prod_{j=1}^d u_j^{k_j})^z \phi(g(\gamma^{-1}(u)))\rho(g(\gamma^{-1}(u)))b(u)\prod_{j=1}^d |u_j|^{h_j} du
\end{aligned}$$

Define $V^0 = \gamma(g^{-1}(U^0))$. Since $V^0 \subset \{u \in \mathbb{R}^d : \prod_{j=1}^d u_j^{k_j} = 0\}$, V^0 is also a null set, so we can add it back into the integral. Also since S^z is entire, we can drop it. It remains to consider

$$\int_{V\cup V^0} (\prod_{j=1}^d u_j^{k_j})^z \phi(g(\gamma^{-1}(u)))\rho(g(\gamma^{-1}(u)))b(u)\prod_{j=1}^d |u_j|^{h_j} du$$

Since $\operatorname{supp}\rho \subset U$, by Hironaka's theorem $g^{-1}(\operatorname{supp}\rho)$ is compact. Then since γ^{-1} is continuous, $\operatorname{supp}\rho \circ g \circ \gamma^{-1}$ is compact. And because $V \cup V^0 = \gamma(g^{-1}(U))$, $\operatorname{supp}\rho \circ g \circ \gamma^{-1} \subset V \cup V^0$. Additionally, $\phi(g(\gamma^{-1}(u)))\rho(g(\gamma^{-1}(u)))b(u)$ is smooth. With these conditions met, we conclude by Corollary 3.4. □

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