MULTIVARIATE ZETA FUNCTIONS, LAPLACE INTEGRALS, AND SINGULAR LEARNING THEORY

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ABSTRACT

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Motivated by questions from singular learning theory, this thesis studies Laplace integrals of the form

$$M_\sigma = \int_W \exp\Big(-\sum_{i=1}^k t_{\sigma i} f_i(w)\Big)\phi(w)\,dw,$$

where the functions f_i are non-negative and analytic up to smooth units, and the weights $t_{\sigma i} = e^{\sigma\theta_i}$ may grow at different rates. To encode interactions between the local behaviors of each f_i around their zero sets, we use the multivariate zeta function

$$\zeta(z) = \int_W \prod_{i=1}^k f_i(w)^{z_i}\phi(w)\,dw.$$

We prove that ζ admits a meromorphic continuation to $\mathbb{C}^k$ and that its poles lie in a locally finite affine hyperplane arrangement with finitely many normal spaces, under hypotheses that allow the boundary of the domain of integration to be defined by analytic functions, and without requiring the weight ϕ to be compactly supported in the interior.

Using Mellin inversion, a residue theory for locally finite affine hyperplane arrangements, and a graph traversal algorithm for shifting contours across chambers, we show that for every $\eta \in \mathbb{R}$, M_σ can be written as a finite sum of terms of the form

$$p_\iota(\sigma)e^{-\sigma\lambda_\iota},$$

up to $o(e^{-\sigma\eta})$ error. Each $\lambda_\iota > 0$ is the optimal value of a linear program determined by the polar arrangement of ζ and p_ι is a polynomial with degree controlled by the polar order at the corresponding optimal face.

This extends the one function and one corresponding complex variable setting of singular learning theory to several functions and several corresponding complex variables, which provides a framework for studying questions involving multiple loss functions, including transfer learning and regularization.

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1 Introduction: machine learning and singular learning theory

This section is an introduction to machine learning and singular learning theory, aimed at a mathematical audience without a background in machine learning. It is not necessary to read this section to understand the statement or proof of the main theorem, but we recommend reading it for motivation.

1.1 Machine learning

Let X be a set and μ an unknown probability measure on X. Let $W \subseteq \mathbb{R}^d$ be the set of parameters for a statistical model. That is, we choose a family of probability measures $\{\nu_w\}_{w \in W}$ such that μ is absolutely continuous with respect to ν_w. By the Radon-Nikodym theorem, there exists a density function $\frac{d\mu}{d\nu_w} : X \to \mathbb{R}$ such that $\mu(E) = \int_E \frac{d\mu}{d\nu_w}(x) d\nu_w$ for all measurable $E \subseteq X$.

The KL divergence (relative information, relative entropy) from μ to ν_w is defined as

$$\mathrm{KL}(\mu \| \nu_w) := \int_X \log(\frac{d\mu}{d\nu_w}(x)) d\mu.$$

For brevity, write $K(w) := \mathrm{KL}(\mu \| \nu_w)$. This is one of the most fundamental objects of information theory [12]. If the data are modeled by ν_w, then $K(w)$ measures, on average, how much more surprising the data are than under the true measure μ. In particular, $K(w) \geq 0$, with equality if and only if $\nu_w = \mu$. This follows because $t \log t \geq t - 1$, with equality if and only if $t = 1$:

$$\begin{aligned}
&\int_X \log(\frac{d\mu}{d\nu_w}(x)) d\mu \\
&= \int_X \frac{d\mu}{d\nu_w}(x) \log(\frac{d\mu}{d\nu_w}(x)) d\nu_w \\
&\geq \int_X (\frac{d\mu}{d\nu_w}(x) - 1) d\nu_w \qquad \text{with equality if and only if } \frac{d\mu}{d\nu_w}(x) = 1 \text{ almost everywhere} \\
&= \int_X d\mu - \int_X d\nu_w \\
&= 1 - 1 \\
&= 0.
\end{aligned}$$

In a standard machine learning setting, we receive $\{x_i\}_{i=1}^n \subseteq X$ independent samples distributed according to μ. The goal is to learn a model ν_w from the hypothesis class $\{\nu_w\}_{w \in W}$ that is close to μ, as measured by K. A typical hypothesis class may be the set of probability measures that are represented by a fixed neural network architecture with parameters $w \in W = \mathbb{R}^d$.

In practice, we do not know μ, so we cannot directly compute K. Instead, we use the given samples as a proxy for μ. Define the empirical relative information by $K_n(w) = \frac{1}{n} \sum_{i=1}^n \log(\frac{d\mu}{d\nu_w}(x_i))$. Again, because μ is unknown, K_n cannot be computed. However, choose any probability measure ρ such that μ is absolutely continuous with respect to ρ and ρ is absolutely continuous with respect to

each ν_w. Then write, using the chain rule for Radon-Nikodym derivatives,

$$\begin{aligned} K_n(w) &= \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{d\mu}{d\nu_w}(x_i)\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{d\mu}{d\rho}(x_i)\frac{d\rho}{d\nu_w}(x_i)\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{d\mu}{d\rho}(x_i)\right) + \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{d\rho}{d\nu_w}(x_i)\right). \end{aligned}$$

The first term is independent of w, so K_n can be computed up to a constant with respect to w, and $\nabla_w K_n$ can be computed exactly. A typical strategy for machine learning algorithms is to find a parameter $w \in W$ such that $K_n(w)$ is small by running (stochastic) gradient descent on K_n. There is a wide literature about understanding why (stochastic) gradient descent can optimize K_n effectively, even though K_n is typically not convex for multi-layer neural networks [17, 2, 13], but this is not the focus of this thesis.

There is a price for optimizing K_n instead of K: there may be parameters $w \in W$ such that $K_n(w) \ll K(w)$. This is sometimes taken as a formal definition of overfitting to the data.

Before large neural networks ($d \approx n$ or $d \gg n$) demonstrated empirical success, it was commonly believed that large neural networks would catastrophically overfit. Here is one argument for this view. The observed dataset is only finite, so there are many probability measures that assign high probability to the data, but behave very differently on unobserved data. Large neural networks can approximate a wide range of probability measures [5, 38], so it is possible to find many parameters $w \in W$ such that $K_n(w) \ll K(w)$. The tension between this heuristic and the empirical success of large neural networks is another central theme in the literature [38, 6, 27, 37].

One way to mitigate this issue is to take a Bayesian approach. By placing a prior on the parameters $w \in W$, we can constrain how strongly the model can update based on the data and avoid complex hypotheses that happen to fit the data. In a sense, a prior is a definition of hypothesis complexity; this idea is formalized in algorithmic information theory [21].

Precisely, we place a prior density $\phi : W \to \mathbb{R}_{\geq 0}$ on the parameter space, normalized so that

$$\int_W \phi(w)\, dw = 1.$$

After observing independent samples $\{x_i\}_{i=1}^n$ from μ, we update this prior by favoring parameters that fit the data well. This leads to the tempered posterior

$$\phi_{n,\beta}(w) := \frac{e^{-\beta n K_n(w)}\phi(w)}{\int_W e^{-\beta n K_n(w)}\phi(w)\, dw},$$

where $\beta > 0$ is known as the inverse temperature. When $\beta = 1$, this is the usual Bayesian posterior density. If β is large, the model updates more strongly based on the data, and if β is small, the model updates less strongly based on the data. One way to see this is that the tempered posterior is the unique probability measure that minimizes the functional

$$\beta n \int_W K_n(w) d\pi + \mathrm{KL}(\pi \| \Phi),$$

where π ranges over probability measures over W such that π is absolutely continuous with respect to the prior Φ [10, Lemma 1.1.3], and Φ is the probability measure on W with density function ϕ. We can interpret this as saying that $\phi_{n,\beta}$ optimally trades off its error on the observed data and its divergence to the prior, where $\int_W K_n(w)d\pi$ controls the error on the observed data and $\mathrm{KL}(\pi\|\Phi)$ controls the divergence to the prior.

Using $\phi_{n,\beta}$, there are two natural algorithms to predict a new data point x_{n+1}. In Bayes estimation, we average the distributions ν_w against the posterior and use the posterior predictive measure

$$\tau(E) := \int_W \nu_w(E)\phi_{n,\beta}(w)\,dw$$

for measurable $E \subseteq X$. In Gibbs estimation, we first sample w from the tempered posterior and then predict using ν_w.

For Gibbs estimation, the expected prediction error is

$$\int_W K(w)\phi_{n,\beta}(w)\,dw = \frac{\int_W K(w)e^{-\beta n K_n(w)}\phi(w)\,dw}{\int_W e^{-\beta n K_n(w)}\phi(w)\,dw}.$$

We focus on Gibbs estimation because its asymptotic behavior is easier to analyze, though related results also hold for Bayes estimation [35, Theorem 6.10]. One typically wants the expected prediction error to become small quickly as the number of samples grows, so we ask: how quickly does this quantity decay as $n \to \infty$?

1.2 Classical statistical learning theory

We begin with some results from classical statistical learning theory. We assume that W is compact and the closure of an open set in $\mathbb{R}^d$, ϕ is continuous, K is twice continuously differentiable, and there exists a unique $w^* \in W$ such that $K(w^*) = 0$. We further assume that w^* is in the interior of W, $\phi(w^*) > 0$, and $\nabla^2 K(w^*)$ is non-singular. Under certain assumptions that control the random function $K_n - K$ (under fundamental assumptions 3 and 4 in [35] and assuming $\beta n \to \infty$ and $\beta \to 0$ as $n \to \infty$), we can replace K_n with K without changing the asymptotic form. This is not necessary to state similar theorems, but for pedagogical purposes, it will help us understand the impact of the assumptions about the local behavior of K. After replacing K_n with K, the expected prediction error is

$$\frac{\int_W K(w)e^{-\beta n K(w)}\phi(w)dw}{\int_W e^{-\beta n K(w)}\phi(w)dw}.$$

Because of the non-degeneracy assumptions, we can study the numerator and denominator using Laplace's method. The dominant contributions to the integral come from small neighborhoods of w^* where $K \approx 0$. In small neighborhoods, we can approximate K by its second order Taylor expansion at w^*. This results in the expansion [1, 35]:

$$\frac{\int_W K(w)e^{-\beta n K(w)}\phi(w)dw}{\int_W e^{-\beta n K(w)}\phi(w)dw} = \frac{d}{2\beta n} + o(\frac{1}{\beta n}).$$

For clarity, note that one cannot take β large to make the error small; we assumed $\beta \to 0$ as $n \to \infty$ in order to control $K_n - K$.

Now, observe that the asymptotic error is worse for models with high-dimensional parameter spaces. We also have an asymptotic form for the prior probability of sublevel sets $\{K(w) < \epsilon\}$ as $\epsilon \to 0$:

$$\int_{\{K(w)<\epsilon\}} \phi(w)dw = C\epsilon^{d/2} + o(\epsilon^{d/2}),$$

for some $C > 0$. This fact can also be seen by the local behavior of K: again by approximating K by its second order Taylor expansion at w^*, the sublevel sets are approximately ellipsoids with semi-axes proportional to $\sqrt{\epsilon}$. This gives a more intuitive explanation for why large d leads to large asymptotic error. If d is large, the prior probability of the sublevel sets decays quickly. And if the prior probability of the sublevel sets decays quickly, by Bayes' theorem, more information is needed to concentrate the posterior on the small sublevel sets, which leads to greater expected prediction error.

1.3 Singular learning theory

The results from classical statistical learning theory still suggest that large neural networks will perform poorly. However, multi-layer neural networks typically violate the assumptions of classical statistical learning theory. They can have non-unique minimizers and degenerate Hessians, so the results above are misleading for understanding neural networks [33, 34, 36].

The key idea of Watanabe's singular learning theory [35] is to drop the assumptions that there is a unique minimizer and the Hessian is non-singular, and instead assume that K is an analytic function. Thus we permit the zero set to be an analytic variety and permit much more complicated local behavior around singularities in the zero set. Singularities will help us learn faster.

We can no longer apply Laplace's method. As a substitute, we aim to use the local analytic structure of K around its zero set to understand the distribution of small values of K as a random variable under the prior. To this end, define the zeta function of a statistical model:

$$\zeta(z) := \int K(w)^z \phi(w)dw,$$

where $z \in \mathbb{C}$ is such that the integral is well-defined. The positive integer values $\{\zeta(k)\}_{k\in\mathbb{Z}_{\geq 1}}$ are simply the moments of K as a random variable under the prior. Since K is bounded over compact W, its moment generating function is entire, and thus in particular analytic in an open neighborhood of 0, so these moments uniquely determine the distribution of K [8, Theorem 30.1]. In this sense, ζ restricted to positive integers is already a natural and complete summary of the distribution of K. It is then natural to study ζ as a function of a complex variable z, wherever the integral converges.

One can show by differentiating under the integral sign that ζ is holomorphic on the right half-plane. Using either Hironaka's resolution of singularities or Bernstein-Sato polynomials, one can show a remarkable result: ζ meromorphically continues to all of $\mathbb{C}$, and its poles are at negative rational numbers determined by algebraic data coming from either the resolution of singularities or Bernstein-Sato polynomials [3, 4, 7, 14, 20, 30, 35].

Let $\lambda \in \mathbb{R}_{>0}$ be such that $-\lambda$ is the location of largest pole of the zeta function and let m be the order of the pole. λ is known as the real log canonical threshold (RLCT). Then the prior probability

of small sublevel sets satisfies [35, Theorem 7.1]:

$$\int_{\{K(w)<\epsilon\}} \phi(w)dw = C\epsilon^{\lambda}(-\log\epsilon)^{m-1} + o(\epsilon^{\lambda}(-\log\epsilon)^{m-1}),$$

for some $C > 0$. Learning theorists often describe notions of "effective dimensionality" to capture the intuition that only some dimensions of the parameter space matter significantly locally [25, 23]. A desirable property of "effective dimensionality" is that it determines a power-law decay rate for the prior probability of small sublevel sets. We *almost* see this behavior here. A strict power-law decay rate is too restrictive, since logarithmic factors may appear. Nevertheless, this shows that λ generalizes $\frac{d}{2}$ from non-singular models.

Finally, λ determines the asymptotic expected prediction error [35]:

$$\frac{\int_W K(w)e^{-\beta nK(w)}\phi(w)dw}{\int_W e^{-\beta nK(w)}\phi(w)dw} = \frac{\lambda}{\beta n} + o(\frac{1}{\beta n}).$$

λ can be much smaller than $\frac{d}{2}$. Empirically, large neural networks exhibit very small RLCTs; [18] reports values on the order of $0.0001d$ to $0.001d$. Also, if there exists any w^* in the interior of W such that $K(w^*) = 0$ and $\phi(w^*) > 0$, then $\lambda \leq \frac{d}{2}$ [35, Theorem 7.2]. In that sense, classical statistical learning theory assumed the worst case scenario.

Intuitively, SLT shows that a hypothesis class can be globally large in the sense that it can approximate a wide range of probability measures, and yet locally small in the sense that it has highly degenerate singularities and thus concentrates quickly around critical points. This provides a partial explanation for why large neural networks, despite their high-dimensional parameter spaces, can still achieve small expected prediction error on data not seen during training [36].

1.4 Multiple loss functions

To understand model behavior, especially on unseen data, we want to understand the algorithms that the model implements. An interesting hypothesis about singular learning theory, called structural Bayesianism [26], is that at singular critical points, neural networks implement algorithms that exploit structure in the data.

For a concrete example, in [11], the authors trained two-layer neural networks to compute finite group operations. They found through reverse-engineering that the neural networks implement finite group operations via a particular algorithm that leverages representation theory.

For a more speculative example, consider transfer learning, where model developers first train a model on a source dataset, such as general internet text, and then train it on a target dataset, such as International Mathematical Olympiad problems [22]. The goal is to learn foundational capabilities on the source dataset and transfer those skills to the target dataset. From the perspective of structural Bayesianism, one may ask whether interactions between different learned skills are reflected in interactions between singularities of the loss landscapes from each dataset.

Motivated by this possibility, in this thesis, we aim to understand the impact of interactions between singularities of multiple functions that are analytic up to smooth units. This leads to Laplace

integrals of the form

$$\int_W \exp\Big(-\sum_{i=1}^{k} t_i f_i(w)\Big)\phi(w)\,dw,$$

where $f_1, \ldots, f_k$ are non-negative functions that are analytic up to smooth units and the weights $t_i \to \infty$ may grow at different rates. Watanabe's original singular learning theory setting corresponds to the case $k = 1$, where $f_1 = K$ and $t_1 = \beta n$. In the transfer learning setting, we may set $k = 2$, take f_1 to be the error function for the target dataset, take f_2 to be the error function for the source dataset, take n_1 and n_2 to be the number of data points in each dataset, and take β_1 and β_2 to be inverse temperatures for each dataset. Then the expected error for the target dataset is

$$\frac{\int_W f_1(w) e^{-\beta_1 n_1 f_1(w) - \beta_2 n_2 f_2(w)} \phi(w) dw}{\int_W e^{-\beta_1 n_1 f_1(w) - \beta_2 n_2 f_2(w)} \phi(w) dw}.$$

By packaging $f_1\phi$ into a smooth function, we can study the numerator and denominator in the same framework.

Once $k > 1$, the asymptotic behavior is no longer controlled by a one-variable zeta function. Instead, we find that it is governed by interactions between the local analytic structure of the functions f_i around their zero sets, which is encoded by the multivariate zeta function

$$\zeta(z) := \int_W \prod_{i=1}^{k} f_i(w)^{z_i} \phi(w)\,dw.$$

The statistical discussion above serves only as motivation. Starting in Section 2.1, we work in an abstract setting and do not use any probabilistic structure. The next section states the precise assumptions and shows how the locally finite polar affine hyperplane arrangement of the multivariate zeta function determines the asymptotic expansion of the corresponding Laplace integral.

This thesis may also provide a framework for extensions in which the functions f_i are negative or complex, the weights t_i are imaginary or complex, or the functions are random. Additionally, due to an approximation made for computational benefits, the method proposed in [18] to estimate λ over a small open neighborhood of a point w^* relies on an open conjecture which can be reduced to a special case of our setting where $k = 2$ and $f_2(w) = |w - w^*|^2$.

2 Overview and main theorem

2.1 Main setting and main theorem, version 1

Definition 1 (Main Setting)**.**

1. ***Domain.*** *Let $W \subsetneq \mathbb{R}^d$ be nonempty and compact. Assume there is an open neighborhood $X \subseteq \mathbb{R}^d$ of W and there are analytic functions $\pi_1, \dots, \pi_m$ on X such that the interior*

$$W^\circ = \operatorname{int}(W) = \{x \in X : \ \pi_s(x) > 0 \ \forall s \in [1, m]\}$$

is nonempty and $W = \overline{W^\circ}$.

2. ***Weight.*** *Let $\phi : X \to \mathbb{R}$ be smooth.*

3. ***Functions.*** *For each $i \in [1, k]$, let $\widetilde{f}_i : X \to \mathbb{R}$ be analytic and let $\upsilon_i : X \to \mathbb{R}$ be smooth and nowhere-vanishing. Define $f_i : X \to \mathbb{R}$ by*

$$f_i(x) = \upsilon_i(x)\widetilde{f}_i(x) \qquad \text{for } x \in X.$$

Assume further that $f_i(w) \geq 0$ for all $w \in W$ and all $i \in [1, k]$, and that each $\widetilde{f}_i$ is not identically zero on any connected component of X.

4. ***Scales.*** *Let $\theta \in \mathbb{R}^k_{>0}$ and define $t_{\sigma i} := e^{\sigma\theta_i}$ for $\sigma \in \mathbb{R}_{>0}$, $i \in [1, k]$.*

Remark 1. *Item (3) says that each f_i is analytic up to a smooth unit. The point of the factorization $f_i = \upsilon_i \widetilde{f}_i$ is that the local vanishing structure of f_i around its zero set is the same as the local vanishing structure of $\widetilde{f}_i$ around its zero set. Since $\widetilde{f}_i$ is analytic, we can apply tools from algebraic geometry to study $\widetilde{f}_i$. The smooth unit υ_i simply allows us to work with a slightly more general class of functions f_i without changing the essential structure of the problem from the case $\upsilon_i = 1$.*

Remark 2. *By using a smooth partition of unity over a finite open cover of W, our results can be generalized to the case that X is a real analytic manifold and W° is semianalytic. This generalization is taken up in [4] to study the zeta function when $k = 1$. We choose the Euclidean setting to simplify the notation.*

Definition 2 (Laplace integral)**.** *Define*

$$M_\sigma := \int_W e^{-\sum_{i=1}^k t_{\sigma i} f_i(w)} \phi(w)\, dw.$$

We first state a coarse version of the main theorem, which records the shape of the expansion but suppresses the construction of its objects. The rest of this section constructs those objects explicitly. In Subsection 2.4, we state a second version of the main theorem whose objects are defined in terms of the polar arrangement, residue theory, and graph traversal algorithm.

Theorem 1 (Main theorem, version 1)**.** *(→ proof) Let M_σ be as in Definition 2. Then there exist an index set $\mathcal{I}$ and, for each $\iota \in \mathcal{I}$, a polynomial $p_\iota : \mathbb{R} \to \mathbb{R}$ and a scalar $\lambda_\iota \in \mathbb{R}_{>0}$ such that, for every $\eta \in \mathbb{R}$, there is a finitely supported function $\chi_\eta : \mathcal{I} \to \mathbb{Z}$ for which, as $\sigma \to \infty$,*

$$M_\sigma = \sum_{\iota \in \mathcal{I}} \chi_\eta(\iota) e^{-\sigma\lambda_\iota} p_\iota(\sigma) + o(e^{-\sigma\eta}).$$

Remark 3. *If $W \cap \bigcap_{i=1}^k f_i^{-1}(0) = \emptyset$, the theorem is trivial with $\mathcal{I} = \emptyset$. The theorem is also trivial if ϕ is identically zero. Because we do not prove a general criterion ensuring that the first formally admissible term determined by the geometry has a nonzero coefficient, we state the theorem without extra hypotheses and leave such non-triviality questions to future work.*

2.2 Mellin representation

The first step is to relate the Laplace integral to the multivariate zeta function. More precisely, up to choices of convention and with the measure μ as in Definition 3, ζ is the multivariate Mellin transform of μ and M_σ is the multivariate Laplace transform of μ. We can exploit this relationship to represent M_σ as the complex contour integral in Proposition 1.

Lemma 1. *(→ proof) Let I be a countable index set. Let $X \subseteq \mathbb{R}^d$ be open. For each $i \in I$, let $f_i : X \to \mathbb{R}$ be a real analytic function such that f_i is not identically zero on any connected component of X. Then $\bigcup_{i \in I} f_i^{-1}(0)$ has Lebesgue measure 0.*

Definition 3. *Let μ be the finite signed Radon measure on $\mathbb{R}^k$ obtained by pushing forward the signed measure $1_W(w)\phi(w)dw$ through the map $(f_1, \ldots, f_k)$.*

Lemma 2. *(→ proof) Let M_σ be as in Definition 2 and μ be as in Definition 3. Then*

$$M_\sigma = \int_{(0,\infty)^k} e^{-\sum_{i=1}^k t_{\sigma i} x_i} d\mu(x).$$

Definition 4 (Zeta function)**.** *Let μ be as in Definition 3. Let $H^k := \{z \in \mathbb{C}^k : \ \forall i \in [1,k] \ Re(z_i) > 0\}$. Define $\zeta : H^k \to \mathbb{C}$ by*

$$\zeta(z) := \int_{(0,\infty)^k} x^z \, d\mu(x) = \int_{W \setminus \left(W \cap \bigcup_{i=1}^k f_i^{-1}(0)\right)} \prod_{i=1}^k f_i(w)^{z_i} \, \phi(w) \, dw = \int_W \prod_{i=1}^k f_i(w)^{z_i} \, \phi(w) \, dw.$$

In the last integral, we use the convention that $\prod_{i=1}^k f_i(w)^{z_i} = 0$ when some $f_i(w) = 0$, for $z \in H^k$. The zero set of f_i is the same as the zero set of $\widetilde{f_i}$. Lemma 1 applied to $\widetilde{f_1}, \ldots, \widetilde{f_k}$ on X therefore shows that $W \cap \bigcup_{i=1}^k f_i^{-1}(0)$ has Lebesgue measure 0, which justifies the last equality.

We will show that ζ is holomorphic on H^k in Lemma 17 and that ζ meromorphically continues to $\mathbb{C}^k$ in Theorem 9.

Previous literature has shown that similar zeta functions meromorphically continue to $\mathbb{C}^k$ in various settings. In the case $k = 1$, meromorphic continuation was first proven in [4] using resolution of singularities. Also for $k = 1$, [14] does not require the function $f := f_1$ to be non-negative, and instead fixes a branch of the complex logarithm to define f^z. Our proof most closely resembles the proof of the $k = 1$ case in [3, Lemma 7.3] by reducing to a toy zeta function. Cardenal [20, Section 5.1] describes a result in [30] showing that a similar zeta function meromorphically continues to $\mathbb{C}^k$ for $k \geq 1$, where the domain of integration is a smooth complex variety and the weight ϕ is smooth with compact support in the variety.

In certain cases, such as when ϕ is smooth with compact support inside the domain of integration, the zeta function can be studied more easily using Bernstein-Sato polynomials [7], in the case $k = 1$,

or their generalization, Bernstein-Sato ideals [30, 15], in the case $k \geq 1$. If ϕ is smooth with compact support inside the domain of integration, one may use integration by parts to repeatedly apply the defining equation of the Bernstein-Sato polynomial or ideal; the boundary term vanishes because of the support of ϕ. See [35, Section 4.5] for a simple proof of the meromorphic continuation of the zeta function in this case with $k = 1$.

To the best of our knowledge, this thesis is the first work to prove the meromorphic continuation of a similar zeta function that integrates over a domain satisfying our assumptions, with $k \geq 1$, and without assuming ϕ is smooth with compact support inside the domain of integration.

Definition 5 (Linear objectives)**.** *Define $\omega, \omega_\sigma \in (\mathbb{R}^k)^*$ by*

$$\omega(x) := -\sum_{i=1}^{k} x_i \theta_i, \qquad \omega_\sigma(x) := \sigma\omega(x).$$

Thus $e^{-\omega_\sigma(z)} = \prod_{i=1}^{k} t_{\sigma i}^{z_i}$.

The growth weights $t_{\sigma i} = e^{\sigma\theta_i}$ enter our analysis through the linear functional that they define on the Mellin variables. In the main setting, we assumed this form of $t_{\sigma i}$ precisely so that the direction of ω_σ is independent of σ. If future work relaxes this assumption, it may need to carefully adapt the graph traversal algorithm in Section 7.

Definition 6. *Let ζ denote the meromorphic continuation of the function in Definition 4. Define*

$$\gamma(z) := \zeta(z) \prod_{i=1}^{k} \Gamma(-z_i).$$

We also obtain Lemma 3 through Hironaka's theorem.

Lemma 3. *(→ proof) There exists $c_0 \in (-\infty, 0)^k$ for which $\int_{(0,\infty)^k} x^{c_0}\, d|\mu|(x) < \infty$.*

We will use Mellin inversion to prove Proposition 1.

Proposition 1 (Initial contour representation)**.** *(→ proof) Let c_0 be as in Lemma 3, M_σ be as in Definition 2, γ be as in Definition 6, and ω_σ be as in Definition 5. Then*

$$M_\sigma = \frac{1}{(2\pi i)^k} \int_{c_0 + i\mathbb{R}^k} e^{-\omega_\sigma(z)} \gamma(z)\, \Pi_{\mathbb{C}^k}, \qquad \Pi_{\mathbb{C}^k} := dz_1 \wedge \cdots \wedge dz_k. \tag{1}$$

Throughout the rest of the paper, $\Pi_{\mathbb{C}^k}$ denotes this standard holomorphic volume form on $\mathbb{C}^k$.

2.3 Polar affine hyperplane arrangement

In this section we consider the structure of the polar set of ζ. Locally around each point of the zero locus $W \cap \bigcup_{i=1}^{k} f_i^{-1}(0)$, we use Hironaka's theorem to write the analytic factors $\widetilde{f}_i$ and each π_s locally as monomials on a chart of the blow-up, up to analytic units. Consequently, the original functions f_i become monomials times positive smooth units, while the smooth weight is pulled back as a smooth factor. To see the consequence of this in the simplest case, consider $W = [0, 1]^d$, which is defined

by $\pi_s(w) = w_s$ for $s \in [1,d]$ and $\pi_s(w) = 1 - w_{s-d}$ for $s \in [d+1, 2d]$, along with $f_i(w) = \prod_{l=1}^d w_l^{k_{il}}$, and suppose for illustration that the weight also has the monomial form $\phi(w) = \prod_{l=1}^d w_l^{h_l}$. Then

$$\begin{aligned}
\zeta(z) &= \int_W \prod_{i=1}^k f_i(w)^{z_i} \phi(w) dw \\
&= \int_{[0,1]^d} (\prod_{i=1}^k (\prod_{l=1}^d w_l^{k_{il}})^{z_i}) (\prod_{l=1}^d w_l^{h_l}) dw \\
&= \int_{[0,1]^d} \prod_{l=1}^d w_l^{\sum_{i=1}^k k_{il} z_i + h_l} dw \\
&= \frac{1}{\prod_{l=1}^d (\sum_{i=1}^k k_{il} z_i + h_l + 1)},
\end{aligned}$$

and thus we obtain the meromorphic continuation of ζ to $\mathbb{C}^k$ in this case. Observe that the polar set is a union of affine hyperplanes. We generalize this idea.

Definition 7. *Let* $\mathrm{Aff}(\mathbb{R}^k, \mathbb{R})$ *be the set of affine functions* $\mathbb{R}^k \to \mathbb{R}$. *For each* $q \in \mathrm{Aff}(\mathbb{R}^k, \mathbb{R})$, *define*

$$\begin{aligned}
Z(q) &:= \{x \in \mathbb{R}^k : q(x) = 0\} \\
Z_{\mathbb{C}}(q) &:= \{z \in \mathbb{C}^k : q(z) = 0\}
\end{aligned}$$

Remark 4. *In this work, we neglect to distinguish between analytic functions* $\mathbb{R}^k \to \mathbb{R}$ *and their complexifications. For instance,* $Z_{\mathbb{C}}(q)$ *can more formally be interpreted as*

$$Z_{\mathbb{C}}(q) := \{z \in \mathbb{C}^k : (q \otimes_{\mathbb{R}} 1_{\mathbb{C}})(z) = 0\}.$$

Definition 8. *Let* $\mathcal{P} \subsetneq \mathbb{R}^k$ *be a real affine hyperplane arrangement. Define*

$$\mathfrak{Q}(\mathcal{P}) := \{q \in \mathrm{Aff}(\mathbb{R}^k, \mathbb{R}) \setminus \mathrm{Con}(\mathbb{R}^k, \mathbb{R}) : Z(q) \subseteq \mathcal{P}\},$$

where $\mathrm{Aff}(\mathbb{R}^k, \mathbb{R})$ *is the set of affine functions* $\mathbb{R}^k \to \mathbb{R}$ *and* $\mathrm{Con}(\mathbb{R}^k, \mathbb{R})$ *is the set of constant functions* $\mathbb{R}^k \to \mathbb{R}$.

Definition 9. *Let* $\mathcal{P} \subsetneq \mathbb{R}^k$ *be a real affine hyperplane arrangement. Let* $\mathfrak{Q}(\mathcal{P})$ *be as in Definition 8. For each open* $U \subseteq \mathbb{C}^k$, *define*

$$R(U, \mathcal{P}) := \mathcal{O}(U)[q^{-1} : q \in \mathfrak{Q}(\mathcal{P})],$$

as the $\mathbb{C}$*-algebra of meromorphic functions on* U *with each* $q \in \mathfrak{Q}(\mathcal{P})$ *inverted. Next, for each open* $U \subseteq \mathbb{C}^k$, *define*

$$\widehat{R}(U, \mathcal{P}) := \{f \in \mathcal{M}(U) : \forall \text{ pre-compact } \widetilde{U} \subseteq U, f|_{\widetilde{U}} \in R(\widetilde{U}, \mathcal{P})\},$$

as the $\mathbb{C}$*-algebra of meromorphic functions on* U *whose restriction to any open pre-compact* $\widetilde{U} \subseteq U$ *is in* $R(\widetilde{U}, \mathcal{P})$.

An element of the localization $R(\mathbb{C}^k, \mathcal{P})$ only has finitely many affine forms in the denominator over $\mathbb{C}^k$, but $\widehat{R}(\mathbb{C}^k, \mathcal{P})$ allows us to consider meromorphic functions with infinitely many affine forms that eventually appear in the denominator over large enough sets.

We started with real affine hyperplane arrangements $\mathcal{P} \subsetneq \mathbb{R}^k$, but we could have allowed complex affine hyperplane arrangements $\mathcal{P} \subsetneq \mathbb{C}^k$, defined $\mathfrak{Q}(\mathcal{P}) := \{q \in \mathrm{Aff}(\mathbb{C}^k, \mathbb{C}) \setminus \mathrm{Con}(\mathbb{C}^k, \mathbb{C}) : Z(q) \subseteq \mathcal{P}\}$, and defined $R(U, \mathcal{P})$ and $\widehat{R}(U, \mathcal{P})$ with respect to this more general definition of $\mathfrak{Q}(\mathcal{P})$. In this work, we will at times consider complex affine hyperplane arrangements, but they are always induced from real affine hyperplane arrangements in the sense of Definition 10. For clarity, we use definitions that are based on the real affine hyperplane arrangements.

Definition 10. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a real affine hyperplane arrangement. Let $\mathfrak{Q}(\mathcal{P})$ be as in Definition 8. Define*

$$\mathcal{P}_{\mathbb{C}} := \bigcup_{q \in \mathfrak{Q}(\mathcal{P})} Z_{\mathbb{C}}(q).$$

Theorem 2. *(→ proof) Let ζ be as in Definition 4. Then ζ admits a meromorphic continuation to $\mathbb{C}^k$.*

Moreover, there exist a finite set $\mathcal{J}$ and integer data $(k_{jil})_{j \in \mathcal{J}, i \in [1,k], l \in [1,d]}$ and $(h_{jl})_{j \in \mathcal{J}, l \in [1,d]}$ such that, if

$$\mathcal{P} := \bigcup_{j \in \mathcal{J}, l \in [1,d], \beta \in \mathbb{Z}_{\geq 0}} \Big\{ z \in \mathbb{R}^k : \sum_{i=1}^{k} k_{jil} z_i + h_{jl} + \beta + 1 = 0 \Big\},$$

then $\zeta \in \widehat{R}(\mathbb{C}^k, \mathcal{P})$. The set $\mathcal{P}$ is a possibly empty locally finite affine hyperplane arrangement with finitely many normal spaces.

Theorem 9 gives a more precise formula for the meromorphic continuation of ζ, but we omit that formula here to keep the overview readable.

Proposition 2. *(→ proof) Let $\mathcal{P}_1 \subsetneq \mathbb{R}^k$ be the locally finite real affine hyperplane arrangement as in Theorem 2. Let $\mathcal{P}_2 \subsetneq \mathbb{R}^k$ be the locally finite real affine hyperplane arrangement defined by*

$$\mathcal{P}_2 := \bigcup_{i \in [1,k], \beta \in \mathbb{Z}_{\geq 0}} \{ z \in \mathbb{R}^k : z_i = \beta \}.$$

Define

$$\mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2.$$

Let γ be as in Definition 6. Then $\gamma \in \widehat{R}(\mathbb{C}^k, \mathcal{P})$. Moreover, $\mathcal{P}$ is a locally finite affine hyperplane arrangement with finitely many normal spaces.

$\mathcal{P}$ from Proposition 2 is the main locally finite affine hyperplane arrangement that we consider in this work.

Definition 11 ($\Omega(\mathcal{P})$ and chambers)**.** *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\Omega(\mathcal{P})$ denote the set of connected components of $\mathbb{R}^k \setminus \mathcal{P}$, which we refer to as chambers.*

Remark 5. *Throughout, by a face of a locally finite real affine hyperplane arrangement $\mathcal{P}$ we mean a closed face of the cell decomposition of $\mathbb{R}^k$ induced by $\mathcal{P}$. In particular, if $\mathfrak{c} \in \Omega(\mathcal{P})$, then $\overline{\mathfrak{c}}$ is a face of $\mathcal{P}$ of codimension 0. Whenever we need a boundary face rather than a chamber closure, we will explicitly assume positive codimension.*

2.4 Main theorem, version 2

We now state an explicit version of the main theorem in which every object appearing in the asymptotic expansion is named.

Theorem 3 (Main theorem, version 2)**.** *(→ proof)*

Fix the following data:

1. *Let $c_0 \in (-\infty, 0)^k$ be as in Lemma 3.*
2. *Let $\mathcal{P}$ be the locally finite real affine hyperplane arrangement defined in Proposition 2.*
3. *Let $\Omega(\mathcal{P})$ be as in Definition 11.*
4. *Let $\mathfrak{c}_0 \in \Omega(\mathcal{P})$ be the chamber containing c_0.*
5. *Let $G_\omega(\mathcal{P})$ be the directed graph in Definition 26 whose vertices are $\Omega(\mathcal{P})$. Each vertex has finitely many edges.*
6. *Let $\lambda(\mathfrak{c}, \omega) \in \mathbb{R} \cup \{\infty\}$ be as in Definition 12.*
7. *Let $p_{\mathfrak{c},\omega}$ be the polynomial in Definition 30.*

For every $L \in \mathbb{Z}_{\geq 1}$, with $\mathbb{P}_j(\mathfrak{c}_0, G_\omega(\mathcal{P}))$ denoting the set of paths in $G_\omega(\mathcal{P})$ of length j starting at $\mathfrak{c}_0$, with $\mathbb{P}_0(\mathfrak{c}_0, G_\omega(\mathcal{P})) := \{(\mathfrak{c}_0)\}$, and with path signs as in Definition 27, one has

$$\begin{aligned} M_\sigma = &\sum_{j=0}^{L-1} \sum_{(\mathfrak{c}_0,\ldots,\mathfrak{c}_j) \in \mathbb{P}_j(\mathfrak{c}_0, G_\omega(\mathcal{P}))} \operatorname{sign}(\mathfrak{c}_0, \ldots, \mathfrak{c}_j) I_\sigma(\mathfrak{c}_j, \omega) \\ &+ \sum_{(\mathfrak{c}_0,\ldots,\mathfrak{c}_L) \in \mathbb{P}_L(\mathfrak{c}_0, G_\omega(\mathcal{P}))} \operatorname{sign}(\mathfrak{c}_0, \ldots, \mathfrak{c}_L) J_\sigma(\mathfrak{c}_L, \omega), \end{aligned}$$

where

$$I_\sigma(\mathfrak{c}, \omega) := \begin{cases} e^{-\sigma\lambda(\mathfrak{c},\omega)} p_{\mathfrak{c},\omega}(\sigma) & : \lambda(\mathfrak{c}, \omega) < \infty \\ 0 & : \lambda(\mathfrak{c}, \omega) = \infty \end{cases}$$

$$J_\sigma(\mathfrak{c}, \omega) := \frac{1}{(2\pi i)^k} \int_{c + i\mathbb{R}^k} e^{-\omega_\sigma(z)} \gamma(z)\, \Pi_{\mathbb{C}^k} \qquad c \in \mathfrak{c}.$$

The quantity $J_\sigma(\mathfrak{c}, \omega)$ is independent of the choice of $c \in \mathfrak{c}$. Moreover, for every $L \in \mathbb{Z}_{\geq 1}$, every path $(\mathfrak{c}_0, \ldots, \mathfrak{c}_L) \in \mathbb{P}_L(\mathfrak{c}_0, G_\omega(\mathcal{P}))$, and every $0 \leq j \leq L$, one has $\lambda(\mathfrak{c}_j, \omega) \in \mathbb{R}_{>0} \cup \{\infty\}$. Finally, for every $\eta \in \mathbb{R}$, there exists $L_0 \in \mathbb{Z}_{\geq 1}$ such that for all $L \geq L_0$ and all paths $(\mathfrak{c}_0, \ldots, \mathfrak{c}_L) \in \mathbb{P}_L(\mathfrak{c}_0, G_\omega(\mathcal{P}))$, as $\sigma \to \infty$,

$$J_\sigma(\mathfrak{c}_L, \omega) \in o(e^{-\sigma\eta}).$$

Theorem 3 is a specialization of a more general traversal theorem that may be of independent interest. The fully general statement appears later as Theorem 12 and is formulated using the function space $\Theta(\mathbb{C}^k, \mathcal{P})$ from Definition 46. Since verifying $f \in \Theta(\mathbb{C}^k, \mathcal{P})$ can be awkward in practice, we first record an easier-to-apply version with a stronger explicit decay hypothesis.

Theorem 4 (Traversal formula under explicit decay hypothesis)**.** *(→ proof) Assume the following data and hypotheses:*

1. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement.*
2. *Let $\Omega(\mathcal{P})$ be the set of chambers in Definition 11.*
3. *Let $\mathfrak{c}_0 \in \Omega(\mathcal{P})$ be a chamber.*
4. *Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$.*
5. *Let $G_\omega(\mathcal{P})$ be the directed graph in Definition 26 whose vertices are $\Omega(\mathcal{P})$. Each vertex has finitely many edges.*
6. *Let $\lambda(\mathfrak{c}, \omega) \in \mathbb{R} \cup \{\infty\}$ be as in Definition 12.*
7. *Let $f \in \widehat{R}(\mathbb{C}^k, \mathcal{P})$, where $\widehat{R}(\mathbb{C}^k, \mathcal{P})$ is the $\mathbb{C}$-algebra of functions in Definition 9.*
8. *Assume that for all $E \subseteq \mathbb{C}^k$ such that $Re(E)$ is compact in the topology of $\mathbb{R}^k$ and*

$$\operatorname{dist}(E, \mathcal{P}_\mathbb{C}) := \inf\{|x - y| : x \in E,\ y \in \mathcal{P}_\mathbb{C}\} > 0,$$

there is $\delta > 0$ such that

$$\sup_{z \in E} (1 + |Im(z)|)^{k+\delta} |f(z)| < \infty.$$

For every $L \in \mathbb{Z}_{\geq 1}$, with $\mathbb{P}_j(\mathfrak{c}_0, G_\omega(\mathcal{P}))$ denoting the set of paths in $G_\omega(\mathcal{P})$ of length j starting at $\mathfrak{c}_0$, with $\mathbb{P}_0(\mathfrak{c}_0, G_\omega(\mathcal{P})) := \{(\mathfrak{c}_0)\}$, and with path signs as in Definition 27, one has

$$\begin{aligned} J(\mathfrak{c}_0, \omega) = &\sum_{j=0}^{L-1} \sum_{(\mathfrak{c}_0, \ldots, \mathfrak{c}_j) \in \mathbb{P}_j(\mathfrak{c}_0, G_\omega(\mathcal{P}))} \operatorname{sign}(\mathfrak{c}_0, \ldots, \mathfrak{c}_j) I(\mathfrak{c}_j, \omega) \\ &+ \sum_{(\mathfrak{c}_0, \ldots, \mathfrak{c}_L) \in \mathbb{P}_L(\mathfrak{c}_0, G_\omega(\mathcal{P}))} \operatorname{sign}(\mathfrak{c}_0, \ldots, \mathfrak{c}_L) J(\mathfrak{c}_L, \omega), \end{aligned}$$

where

$$I(\mathfrak{c}, \omega) := \begin{cases} \frac{1}{(2\pi i)^{\dim(F(\mathfrak{c},\omega))}} \int_{c + i\,\mathrm{lin}(F(\mathfrak{c},\omega))} \operatorname{Res}_{FL(\mathfrak{c},\omega)}(f\, \Pi_{\mathbb{C}^k}) \qquad c \in \operatorname{relint}(F(\mathfrak{c}, \omega)) & : \lambda(\mathfrak{c}, \omega) < \infty \\ 0 & : \lambda(\mathfrak{c}, \omega) = \infty \end{cases}$$

$$J(\mathfrak{c}, \omega) := \frac{1}{(2\pi i)^k} \int_{c + i\mathbb{R}^k} f\, \Pi_{\mathbb{C}^k} \qquad c \in \mathfrak{c}.$$

Here $\operatorname{Res}_{FL(\mathfrak{c},\omega)}$ is the affine residue of Definition 42, and I and J are independent of c.

The graph $G_\omega(\mathcal{P})$, the value $\lambda(\mathfrak{c}, \omega)$, the face $F(\mathfrak{c}, \omega)$, the flag $FL(\mathfrak{c}, \omega)$, the residue $\operatorname{Res}_{FL(\mathfrak{c},\omega)}$, the path signs, and the polynomial $p_{\mathfrak{c},\omega}$ have not yet been defined. The residue theory is introduced

later; the rest of this section constructs the other objects, and at the same time, explains the proof strategy for both the main theorem and the general traversal formula. The argument has three main steps.

1. Starting from the contour formula of Proposition 1, we move the contour across faces of $\mathcal{P}$. A residue theory turns each contour crossing into a jump formula, and the residues produce the dominant terms $e^{-\sigma\lambda(\mathfrak{c},\omega)}p_{\mathfrak{c},\omega}(\sigma)$.

2. To organize which faces to cross, we assign to each chamber $\mathfrak{c} \in \Omega(\mathcal{P})$ a preferred face $F(\mathfrak{c},\omega)$, a flag $FL(\mathfrak{c},\omega)$ with base $F(\mathfrak{c},\omega)$, and a collection of target chambers $\mathfrak{b}_{FL(\mathfrak{c},\omega)}(\alpha) \in \Omega(\mathcal{P})$ across that face, indexed by $\alpha \in \{\pm 1\}^r \setminus \{1\}^r$. Here 1^r corresponds to the original chamber $\mathfrak{c}$.

3. We then show that repeated application of this rule eventually reaches chambers $\mathfrak{c}$ for which $\sup_{x\in\overline{\mathfrak{c}}}\omega(x)$ is arbitrarily large. Choosing $c \in \mathfrak{c}$ with $\omega(c)$ close to this supremum makes the remainder terms smaller than any prescribed exponential scale in σ:

$$\frac{1}{(2\pi i)^k}\int_{c+i\mathbb{R}^k} e^{-\omega_\sigma(z)}\gamma(z)\Pi_{\mathbb{C}^k} \in o(e^{-\sigma\omega(c)}).$$

2.5 Chamber to face

In the next four subsections, we work with an abstract locally finite affine hyperplane arrangement $\mathcal{P} \subsetneq \mathbb{R}^k$. For the main theorem, however, one may keep in mind the specific arrangement from Proposition 2, which is locally finite and has finitely many normal spaces.

When a chamber $\mathfrak{c}$ appears in the asymptotic formula, the size of its remainder term is governed by the value of the objective ω on a point of $\mathfrak{c}$: larger values of ω give stronger exponential decay. Since we may choose any point in the chamber, the best decay available from $\mathfrak{c}$ is controlled by $\sup_{x\in\overline{\mathfrak{c}}}\omega(x)$. Our goal is therefore to construct a rule for moving between chambers so that this supremum eventually becomes as large as we wish.

The first step is to maximize ω on $\overline{\mathfrak{c}}$. If the supremum is infinite, then the chamber already contributes negligibly and no further contour shift is needed. If the supremum is finite, the set of maximizers is a face of $\overline{\mathfrak{c}}$, and that is the natural face across which to move the contour.

Definition 12 (Linear program)**.** *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $\mathfrak{c} \in \Omega(\mathcal{P})$. Define the linear program*

$$\lambda(\mathfrak{c},\omega) := \sup_{x\in\overline{\mathfrak{c}}}\omega(x).$$

Definition 13 (Optimal face)**.** *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $\mathfrak{c} \in \Omega(\mathcal{P})$ be such that $\lambda(\mathfrak{c},\omega) < \infty$. Define the optimal face of $\mathfrak{c}$ by*

$$F(\mathfrak{c},\omega) := \{x \in \overline{\mathfrak{c}} : \omega(x) = \lambda(\mathfrak{c},\omega)\} \subseteq \overline{\mathfrak{c}} \cap \mathcal{P}.$$

The fact that $F(\mathfrak{c},\omega) \subseteq \mathcal{P}$ follows from the fact that ω has non-vanishing gradient.

2.6 Chamber and face to flag

Let $\mathfrak{c}_0$ be a starting chamber and $F(\mathfrak{c}_0, \omega)$ be the optimal face of $\mathfrak{c}_0$. Crossing a face of codimension r produces a jump formula involving 2^r chambers that share that face, including the starting chamber. If exactly 2^r chambers meet along the face, there is no ambiguity. If more than 2^r chambers meet there, then we must choose $2^r - 1$ target chambers in a way compatible with the jump formula. A flag of faces as in Definition 14 is the device that records this choice. Definition 24 constructs the target chambers determined by a choice of flag.

The difficulty is that chambers meeting along $F(\mathfrak{c}, \omega)$ may share the same optimal value. A naive choice of target chambers could therefore create cycles and prevent the traversal from reaching chambers with larger optimal value. Our construction does allow ties, but chooses target chambers in a way that still forces eventual progress.

Figure 1 shows a two-dimensional example with hyperplanes $x_2 = 0$, $x_1 = 0$, $x_2 = -\frac{1}{\sqrt{3}}x_1$, $x_2 = -x_1$, and $x_2 = -\sqrt{3}x_1$. The dotted line represents the objective $\omega(x_1, x_2) = x_1 + x_2$, where the value can be computed in the picture by projecting onto the dotted line. Chambers with optimal value $\lambda(\mathfrak{c}, \omega) = 0$ are shaded light blue, while chambers with optimal value $\lambda(\mathfrak{c}, \omega) > 0$ are shaded light orange. The lower-left dot and darker blue shading mark the starting chamber $\mathfrak{c}_0$.

The figure shows the paths that our algorithm takes from $\mathfrak{c}_0$ to chambers with optimal value $\lambda(\mathfrak{c}, \omega) > 0$. The first step crosses the optimal face of $\mathfrak{c}_0$ at the origin, which has codimension 2. The jump formula therefore lands in $2^2 - 1 = 3$ new chambers, each containing the origin in its closure. Some of these target chambers already satisfy $\lambda(\mathfrak{c}, \omega) > 0$ and thus leave the local picture to cross faces elsewhere. One target chamber still has optimal value 0 and the same optimal face at the origin, so we repeat the procedure. Again, one chamber with optimal value 0 remains, but now its optimal face, $\{(x_1, x_2) : x_2 = -x_1, x_1 \leq 0\}$, has codimension 1. Crossing that face yields a single new chamber, since $2^1 - 1 = 1$, and this chamber has $\lambda(\mathfrak{c}, \omega) > 0$. At that point, the local traversal is complete.

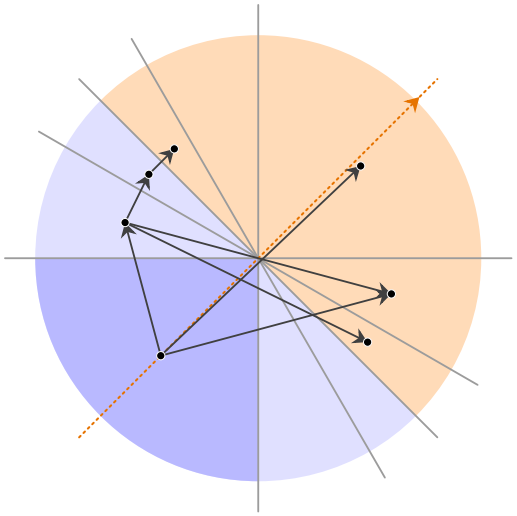


Figure 1: A two-dimensional arrangement showing the paths from the starting chamber to chambers with larger optimal value.

Definition 14 (Flag of faces)**.** *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $\mathfrak{c} \in \Omega(\mathcal{P})$ be such that $\lambda(\mathfrak{c}, \omega) < \infty$. Let F be any face of $\mathcal{P}$ contained in $\overline{\mathfrak{c}}$ with codimension $r \geq 0$. A flag of faces with base F is a sequence of faces $F_0, \ldots, F_r$ such that*

$$F_r = F, \qquad F_i \subsetneq F_{i-1} \ \forall i \in [1, r], \qquad \operatorname{codim}(F_i) = i.$$

We will make a choice of flag with base F. Although we will later specialize to $F = F(\mathfrak{c}, \omega)$, we opt to work at this level of abstraction.

Definition 15. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let F be a face of $\mathcal{P}$ of positive codimension. Define $\Xi(F)$ as the set of faces of codimension $\operatorname{codim}(F) - 1$ containing F.*

Proposition 3 (Generator)**.** *(→ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let F be a face of $\mathcal{P}$ of positive codimension. Let $F' \in \Xi(F)$ and let $u \in \operatorname{lin}(F') \cap \operatorname{lin}(F)^\perp$ be such that $|u| = 1$. The following are equivalent:*

1. *Every $z \in F'$ can be uniquely written as $z = x + tu$ for some $x \in \operatorname{aff}(F)$ and $t \in \mathbb{R}_{\geq 0}$.*
2. *Every $z \in F' \setminus F$ can be uniquely written as $z = x + tu$ for some $x \in \operatorname{aff}(F)$ and $t \in \mathbb{R}_{>0}$.*
3. *For all $z \in F'$ and $x \in \operatorname{aff}(F)$, $\langle z - x, u \rangle \geq 0$.*
4. *For all $z \in F'$, there exists $x \in \operatorname{aff}(F)$ such that $\langle z - x, u \rangle \geq 0$.*
5. *For all $x \in \operatorname{relint}(F)$, there exists $\delta > 0$ such that for all $t \in (0, \delta)$, $x + tu \in \operatorname{relint}(F')$.*
6. *There exists $x \in \operatorname{aff}(F)$, $t \in \mathbb{R}_{>0}$, and $z \in F'$ such that $z = x + tu$.*

There exists a unique $u \in \operatorname{lin}(F') \cap \operatorname{lin}(F)^\perp$ such that $|u| = 1$ and each condition is satisfied. We say that u is the generator of F' over F.

Definition 16 (Auxiliary objectives)**.** *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $\mathfrak{c} \in \Omega(\mathcal{P})$. Let $F \subsetneq \overline{\mathfrak{c}}$ be a face of $\mathcal{P}$ of positive codimension. For each $F' \in \Xi(F)$, define $\omega(F', F) := \omega(u)$, where u is the generator of F' over F as defined in Proposition 3. Define $\mathfrak{P}(F, \mathfrak{c}, \omega) \in \mathbb{R}$ by*

$$\mathfrak{P}(F, \mathfrak{c}, \omega) := \max_{F' \in \Xi(F), F' \subseteq \overline{\mathfrak{c}}} \omega(F', F)$$

The maximum of the auxiliary objective can be achieved at multiple faces $F' \in \Xi(F)$ with $F' \subseteq \overline{\mathfrak{c}}$. For example, in Figure 1, the optimal face of the starting chamber $\mathfrak{c}_0$ is the origin, and there are two codimension 1 faces $F' \in \Xi(F(\mathfrak{c}_0, \omega))$ with $F' \subseteq \overline{\mathfrak{c}_0}$, each of which achieves the maximum of the auxiliary objective. Thus we must decide on a tie-breaking rule.

Lemma 4 allows us to define a tie-breaking rule by optimizing for a secondary ω'. In fact, we can choose ω' independently of $\mathfrak{c}$ or F.

Lemma 4. *(→ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. There exists a Lebesgue measure zero set $E(\mathcal{P}) \subsetneq (\mathbb{R}^k)^*$ containing 0 such that for all $\omega \in (\mathbb{R}^k)^* \setminus E(\mathcal{P})$*

and every face F of $\mathcal{P}$ of positive codimension, the values $\{\omega(F', F)\}_{F' \in \Xi(F)}$ are pairwise distinct. In particular, for every $\mathfrak{c} \in \Omega(\mathcal{P})$ and every face $F \subsetneq \overline{\mathfrak{c}}$, the maximum in Definition 16 is attained at a unique face $F' \in \Xi(F)$ with $F' \subseteq \overline{\mathfrak{c}}$.

For every locally finite real affine hyperplane arrangement $\mathcal{P}$ considered in the rest of this section, fix once and for all a covector $\omega' \in (\mathbb{R}^k)^* \setminus E(\mathcal{P})$, where $E(\mathcal{P})$ is as in Lemma 4.

Definition 17. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let ω' be the standing auxiliary covector chosen above. Let $\mathfrak{c} \in \Omega(\mathcal{P})$. Let $F \subsetneq \overline{\mathfrak{c}}$ be a face of $\mathcal{P}$. Define $Q(F, \mathfrak{c}, \omega)$ as the unique face $F' \in \Xi(F)$ with $F' \subseteq \overline{\mathfrak{c}}$ that is optimal for the lexicographically ordered objective*

$$[\omega(F', F), \omega'(F', F)].$$

Definition 18. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $\mathfrak{c} \in \Omega(\mathcal{P})$ be such that $\lambda(\mathfrak{c}, \omega) < \infty$. Let $r = \operatorname{codim}(F(\mathfrak{c}, \omega))$. Inductively define for $i \in [1, r]$,*

$$F_r(\mathfrak{c}, \omega) := F(\mathfrak{c}, \omega)$$
$$F_{i-1}(\mathfrak{c}, \omega) := Q(F_i(\mathfrak{c}, \omega), \mathfrak{c}, \omega).$$

This defines a flag of faces

$$F(\mathfrak{c}, \omega) = F_r(\mathfrak{c}, \omega) \subsetneq \ldots \subsetneq F_0(\mathfrak{c}, \omega) = \overline{\mathfrak{c}}.$$

We denote this flag of faces by $FL(\mathfrak{c}, \omega)$.

A choice of flag of faces also determines a signed orthonormal basis as follows.

Definition 19. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Consider a flag of faces in $\mathcal{P}$ given by faces $F_0, \ldots, F_r$ with base $F = F_r$. For each $i \in [1, r]$, let u_i be the generator of F_{i-1} over F_i, as defined in Proposition 3. Define $u_1, \ldots, u_r$ as the basis for $\operatorname{lin}(F)^\perp$ determined by the flag.*

This is a signed orthonormal basis. Its signs are part of the chamber-labeling convention: perturbing by $+\epsilon_i u_i$ moves from F_i toward the face F_{i-1} in the chosen flag. Whenever a residue construction is attached to a flag of faces, these are the default signs for the transverse flag coordinates.

2.7 Flag to chambers

Once a flag has been chosen, it determines a family of chambers adjacent to the base face, which we characterize in terms of the sign of each $q \in \mathfrak{Q}(\mathcal{P})$.

Definition 20. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\mathfrak{Q}(\mathcal{P})$ be as in Definition 8. For any $q \in \mathfrak{Q}(\mathcal{P})$ and any face F of $\mathcal{P}$, $\operatorname{sign}(q(x)) \in \{\pm 1, 0\}$ is constant over $x \in \operatorname{relint}(F)$. This allows us to define*

$$\operatorname{sign}_F(q) := \operatorname{sign}_{\operatorname{relint}(F)}(q) := \operatorname{sign}(q(x)) \qquad x \in \operatorname{relint}(F).$$

Note that if $\mathfrak{c} \in \Omega(\mathcal{P})$ is a chamber, $\overline{\mathfrak{c}}$ is a face of codimension 0 and $\mathfrak{c} = \operatorname{relint}(\overline{\mathfrak{c}})$. Thus this definition includes

$$\operatorname{sign}_{\mathfrak{c}}(q) := \operatorname{sign}_{\overline{\mathfrak{c}}}(q) = \operatorname{sign}(q(x)) \qquad x \in \mathfrak{c}.$$

Definition 21. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. For any face F of $\mathcal{P}$, define*

$$\mathfrak{Q}(F, \mathcal{P}) := \{q \in \mathfrak{Q}(\mathcal{P}) : F \subseteq Z(q)\}.$$

We write

$$F_{\mathbb{C}} := \operatorname{aff}(F) + i \operatorname{lin}(F) \subseteq \mathbb{C}^k$$

for the complexification of the affine span of F.

Definition 22. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Consider any flag of faces of $\mathcal{P}$ given by faces $F_0, \ldots, F_r$ with base $F = F_r$ and with $F_0 = \overline{\mathfrak{c}}$. Then for all $i \in [1, r]$,*

$$\mathfrak{Q}(F_i, \mathcal{P}) \supsetneq \mathfrak{Q}(F_{i-1}, \mathcal{P}),$$

and

$$\mathfrak{Q}(F_0, \mathcal{P}) = \mathfrak{Q}(\overline{\mathfrak{c}}, \mathcal{P}) = \emptyset,$$

where $\mathfrak{Q}(\cdot, \mathcal{P})$ is as in Definition 21. This determines a partition $\{P_i\}_{i=1}^r$ of $\mathfrak{Q}(F, \mathcal{P}) = \mathfrak{Q}(F_r, \mathcal{P})$ by

$$P_i = \mathfrak{Q}(F_i, \mathcal{P}) \setminus \mathfrak{Q}(F_{i-1}, \mathcal{P}).$$

Definition 23 (Greatest index)**.** *Assume the setting of Definition 22. For any $q \in \mathfrak{Q}(F, \mathcal{P})$, define* $\operatorname{gi}(q)$ *as the unique index such that $q \in P_{\operatorname{gi}(q)}$.*

Proposition 4 lets us translate a flag into a canonical choice of neighboring chambers.

Proposition 4. *($\to$ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\mathfrak{c} \in \Omega(\mathcal{P})$. Let*

$$F = F_r \subsetneq \ldots \subsetneq F_0 = \overline{\mathfrak{c}}$$

be a flag of faces of $\mathcal{P}$ with $\operatorname{codim}(F) = r > 0$*, and denote this flag by FL. For each $\alpha \in \{\pm 1\}^r$, there exists a chamber $\mathfrak{b}_{FL}(\alpha) \in \Omega(\mathcal{P})$ and another flag of faces*

$$F = F_r(\alpha) \subsetneq \ldots \subsetneq F_0(\alpha) = \overline{\mathfrak{b}_{FL}(\alpha)}$$

with $\operatorname{codim}(F_i(\alpha)) = i$ *and with signs*

$$\operatorname{sign}_{F_i(\alpha)}(q) = \begin{cases} 0 & : q \in \mathfrak{Q}(F_i, \mathcal{P}) \\ \alpha_{\operatorname{gi}(q)} \operatorname{sign}_{\mathfrak{c}}(q) & : q \in \mathfrak{Q}(F, \mathcal{P}) \setminus \mathfrak{Q}(F_i, \mathcal{P}) \\ \operatorname{sign}_{\mathfrak{c}}(q) & : q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P}) \end{cases}.$$

In particular, $F_i(1^r) = F_i$.

Definition 24 (Chosen neighboring chambers)**.** *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $\mathfrak{c} \in \Omega(\mathcal{P})$ be such that $\lambda(\mathfrak{c}, \omega) < \infty$. Let $FL(\mathfrak{c}, \omega)$ be as in Definition 18. Apply Proposition 4 to $FL(\mathfrak{c}, \omega)$. Then the chosen neighboring chambers are*

$$\mathfrak{b}_{FL(\mathfrak{c},\omega)}(\alpha), \qquad \alpha \in \{\pm 1\}^{\operatorname{codim}(F(\mathfrak{c},\omega))}.$$

2.8 Graph traversal

Recall that we are trying to choose neighboring chambers to push $\lambda(\mathfrak{c}, \omega)$ as large as desired after sufficiently many steps. We would ideally have $\lambda(\mathfrak{b}_{FL(\mathfrak{c},\omega)}(\alpha), \omega) > \lambda(\mathfrak{c}, \omega)$ for all $\alpha \in \{\pm 1\}^r \setminus \{1\}^r$. However, we can only guarantee that $\lambda(\mathfrak{b}_{FL(\mathfrak{c},\omega)}(\alpha), \omega) \geq \lambda(\mathfrak{c}, \omega)$. To obtain a strict increase in some objective, we incorporate the auxiliary objectives of Definition 16.

Definition 25 (Potential function)**.** *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $\mathfrak{c} \in \Omega(\mathcal{P})$. If $\lambda(\mathfrak{c}, \omega) = \infty$, define*

$$\mathfrak{P}^*(\mathfrak{c}, \omega) := [\infty, \ldots, \infty] \in (\mathbb{R} \cup \{\infty\})^{2(k+1)}.$$

If $\lambda(\mathfrak{c}, \omega) < \infty$, define $\mathfrak{P}^*(\mathfrak{c}, \omega) \in (\mathbb{R} \cup \{\infty\})^{2(k+1)}$ by

$$\begin{aligned}\mathfrak{P}^*(\mathfrak{c}, \omega) :=& [\lambda(\mathfrak{c}, \omega), \dim(F(\mathfrak{c}, \omega)), \mathfrak{P}(F_r(\mathfrak{c}, \omega), \mathfrak{c}, \omega), \mathfrak{P}(F_r(\mathfrak{c}, \omega), \mathfrak{c}, \omega'),\\ &\ldots, \mathfrak{P}(F_1(\mathfrak{c}, \omega), \mathfrak{c}, \omega), \mathfrak{P}(F_1(\mathfrak{c}, \omega), \mathfrak{c}, \omega'), 0, \ldots, 0],\end{aligned}$$

where $\lambda(\mathfrak{c}, \omega)$ is defined by Definition 12, $F_i(\mathfrak{c}, \omega)$ is defined by Definition 18, $\mathfrak{P}(F, \mathfrak{c}, \omega)$ is defined by Definition 16, ω' is as in Definition 17, and zeros are padded so that $\mathfrak{P}^*(\mathfrak{c}, \omega) \in (\mathbb{R} \cup \{\infty\})^{2(k+1)}$. We place the lexicographic total order on $(\mathbb{R} \cup \{\infty\})^{2(k+1)}$.

Proposition 5 (Strict potential increase)**.** *(→ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $\mathfrak{c} \in \Omega(\mathcal{P})$ be such that $\lambda(\mathfrak{c}, \omega) < \infty$. Set $r := \operatorname{codim}(F(\mathfrak{c}, \omega))$ and $FL := FL(\mathfrak{c}, \omega)$. For all $\alpha \in \{\pm 1\}^r \setminus \{1\}^r$,*

$$\mathfrak{P}^*(\mathfrak{b}_{FL}(\alpha), \omega) > \mathfrak{P}^*(\mathfrak{c}, \omega).$$

For each $\mathfrak{c} \in \Omega(\mathcal{P})$ such that $\lambda(\mathfrak{c}, \omega) < \infty$, Definition 18 specifies a choice of flag of faces

$$F(\mathfrak{c}, \omega) = F_r(\mathfrak{c}, \omega) \subsetneq \ldots \subsetneq F_0(\mathfrak{c}, \omega) = \overline{\mathfrak{c}}.$$

Then from this flag, Definition 24 specifies chambers $\mathfrak{b}_{FL(\mathfrak{c},\omega)}(\alpha) \in \Omega(\mathcal{P})$ for each $\alpha \in \{\pm 1\}^r \setminus \{1\}^r$. Together, this gives a choice of multiple new chambers from each chamber. We organize these relationships with a directed graph.

Definition 26 (Graph)**.** *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Define a directed graph $G_\omega(\mathcal{P})$ whose vertices are $\Omega(\mathcal{P})$ as follows. For each chamber $\mathfrak{c} \in \Omega(\mathcal{P})$ with $\lambda(\mathfrak{c}, \omega) < \infty$, set $r = \operatorname{codim}(F(\mathfrak{c}, \omega))$ and $FL := FL(\mathfrak{c}, \omega)$, and add an edge from $\mathfrak{c}$ to $\mathfrak{b}_{FL}(\alpha)$ for each $\alpha \in \{\pm 1\}^r \setminus \{1\}^r$. Additionally, if $\lambda(\mathfrak{c}, \omega) = \infty$, add an edge from $\mathfrak{c}$ to itself.*

Corollary 1 is an immediate consequence of Proposition 5.

Corollary 1 (Acyclicity)**.** *(→ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $G_\omega(\mathcal{P})$ be as in Definition 26. Then $G_\omega(\mathcal{P})$ is acyclic, except for self-cycles.*

The flag of faces in Definition 18 and the chambers in Definition 24 are chosen so that Proposition 5 holds, giving a strict increase in the potential function in Definition 25. A strict increase of any potential function implies Corollary 1, but we choose the potential function of Definition 25 because it will allow us to upgrade Corollary 1 to Proposition 6. Proposition 6 is the important payoff that lets us make remainder terms vanish quickly.

Proposition 6. *(→ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $G_\omega(\mathcal{P})$ be as in Definition 26. Suppose $\mathcal{P}$ has finitely many normal spaces. For any starting chamber $\mathfrak{c}_0 \in \Omega(\mathcal{P})$ and any $\eta \in \mathbb{R}$, there exists $L_0 \in \mathbb{Z}_{\geq 1}$ such that for any path $(\mathfrak{c}_0, \ldots, \mathfrak{c}_L)$ of length $L \geq L_0$ in $G_\omega(\mathcal{P})$, $\lambda(\mathfrak{c}_L, \omega) > \eta$.*

Finally, we attach sign labels to the edges of $G_\omega(\mathcal{P})$, which we will use later.

Definition 27. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $G_\omega(\mathcal{P})$ be as in Definition 26.*

Let $\mathfrak{c} \in \Omega(\mathcal{P})$. If $\lambda(\mathfrak{c}, \omega) = \infty$, define $\operatorname{sign}(\mathfrak{c}, \mathfrak{c}) = 1$.

Otherwise, $\lambda(\mathfrak{c}, \omega) < \infty$. Set $r = \operatorname{codim}(F(\mathfrak{c}, \omega))$ and $FL := FL(\mathfrak{c}, \omega)$. For an edge $(\mathfrak{c}, \mathfrak{b}_{FL}(\alpha))$, define

$$\operatorname{sign}(\mathfrak{c}, \mathfrak{b}_{FL}(\alpha)) := -\prod_{i=1}^{r} \alpha_i.$$

For a path $(\mathfrak{c}_0, \ldots, \mathfrak{c}_L)$ of positive length, define

$$\operatorname{sign}(\mathfrak{c}_0, \ldots, \mathfrak{c}_L) := \prod_{i=0}^{L-1} \operatorname{sign}(\mathfrak{c}_i, \mathfrak{c}_{i+1}),$$

and for a path of length 0, define

$$\operatorname{sign}(\mathfrak{c}_0) := 1.$$

2.9 Residue

We construct a residue theory that depends on a choice of flag of faces. The construction of the residue is not needed to state the main theorem, so we do not define it in this section. Instead, at the end of this section, we define the polynomial $p_{\mathfrak{c},\omega}$, which is derived from our residue theory and appears in the main theorem.

Our residue admits a computational formula that generalizes the classical formula that for a meromorphic function of one variable $f \in \mathcal{M}_a$ such that $f = \frac{g}{(z-a)^d}$ for some g that is holomorphic on an open neighborhood of a,

$$\operatorname{Res}_{z=a} f = \frac{1}{(d-1)!} \partial_z^{d-1} g\big|_{z=a}.$$

We perform a triangular change of variables adapted to the choice of flag so that the pullback of a meromorphic function has finitely many negative terms in its Laurent series. Then Proposition 7 shows that, after multiplying by a monomial, the pullback of the meromorphic function becomes holomorphic, which can then be differentiated to extract the residue. This is a higher-dimensional analogue of clearing a finite-order pole in one variable.

Definition 28. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let F be a face of $\mathcal{P}$ of positive codimension and FL be a flag of faces of $\mathcal{P}$ with base F. Set $H := \operatorname{lin}(F)^\perp$ and $H_{\mathbb{C}} := H + iH$. Let $u_1, \ldots, u_r$ be the basis of H determined by FL in Definition 19. Let $u_{1,H}^*, \ldots, u_{r,H}^* \in H^*$ be the dual basis. Write $u_i^* \in (\mathbb{R}^k)^*$ for the real covector obtained from $u_{i,H}^*$ by*

extending it by zero on $\lin(F)$*, and use the same symbol for its* $\mathbb{C}$*-linear extension to* $\mathbb{C}^k$*. Let* b_i *be the constant value of* u_i^* *on* F*, and set* $\rho_i^{FL} := u_i^* - b_i$*. Since* $F_{\mathbb{C}} = \aff(F) + i\lin(F)$*, this extension satisfies* $u_i^*(y) = b_i$ *and* $\rho_i^{FL}(y) = 0$ *for all* $y \in F_{\mathbb{C}}$*.*

Let $x_1, \dots, x_r$ *be the standard basis vectors of* $\mathbb{C}^r$*. Let* $x_1^*, \dots, x_r^* \in (\mathbb{C}^r)^*$ *be the dual basis. Define* $b : \mathbb{C}^r \times F_{\mathbb{C}} \to \mathbb{C}^k$ *by* $b(x, y) := u(x) + y$*, where* $u(x) = \sum_{i=1}^r \big(\prod_{j=i}^r x_j^*(x)\big) u_i$*.*

Let $A = (\mathbb{C}^r \setminus \bigcup_{i=1}^r Z(x_i^*)) \times (\relint(F) + i\lin(F))$ *and*

$$B = \{u + y : u \in H_{\mathbb{C}},\ y \in \relint(F) + i\lin(F),\ \prod_{i=1}^r u_i^*(u) \neq 0\}.$$

Then the restriction $a : A \to B$ *defined by* $a(x, y) := b(x, y)$ *is an analytic isomorphism with inverse* $a^{-1}(u + y) = (u_r^*(u)x_r + \sum_{i=1}^{r-1} \frac{u_i^*(u)}{u_{i+1}^*(u)} x_i, y)$*.*

Lemma 5. *(→ proof) Let* $\mathcal{P} \subsetneq \mathbb{R}^k$ *be a locally finite real affine hyperplane arrangement. Let* $U \subseteq \mathbb{C}^k$ *be open. Let* $\gamma \in R(U, \mathcal{P})$*. Let*

$$\operatorname{div}(\gamma)_- = \sum_{i=1}^n e_i H_i,$$

where $e_1, \dots, e_n \in \mathbb{Z}_{>0}$ *and* $H_1, \dots, H_n \subsetneq \mathbb{C}^k$ *are the complexifications of real affine hyperplanes in* $\mathcal{P}$ *and* $H_1, \dots, H_n$ *intersect* U*. Let* $q_1, \dots, q_n \in \Aff(\mathbb{R}^k, \mathbb{R})$ *be such that* $Z_{\mathbb{C}}(q_i) = H_i$*. Let* $h := \prod_{i=1}^n q_i^{e_i}$*. Then there exists* $g \in \mathcal{O}(U)$ *such that*

$$\gamma = \frac{g}{h},$$

and g *does not vanish identically on any of* $H_1, \dots, H_n$*.*

Definition 29. *Assume the setting of Lemma 5. Fix a flag* $FL = (F_r, \dots, F_0)$ *as in Definition 22, with base* $F = F_r$ *and* $r = \codim(F)$*. Suppose* $q_i \in \mathfrak{Q}(F, \mathcal{P})$ *for all* $i \in [1, n]$*. Let* gi *be the greatest-index function determined by this flag in Definition 23. For* $1 \leq j \leq r$*, define*

$$d_j := \sum_{i=1}^n e_i 1_{\mathrm{gi}(q_i) \leq j}.$$

Also define

$$D_j := \max\{d_j, j\}.$$

Proposition 7. *(→ proof) Let* $\mathcal{P} \subsetneq \mathbb{R}^k$ *be a locally finite real affine hyperplane arrangement. Let* F *be a face of* $\mathcal{P}$ *of positive codimension, and let* $FL = (F_r, \dots, F_0)$ *be a flag as in Definition 22, with base* $F = F_r$ *and* $r = \codim(F)$*. Let* U *be an open neighborhood of* $\relint(F) + i\lin(F)$ *such that* $U \cap Z_{\mathbb{C}}(q) = \emptyset$ *for every* $q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})$*. Let* $\gamma \in R(U, \mathcal{P})$*. Let* $d_1, \dots, d_r$ *be as in Definition 29, for this* F*,* FL*, and the representation of* γ *in Lemma 5. Let* $a : A \to B$ *be the analytic isomorphism as in Definition 28, for the same flag* FL*. Define* $\widetilde{\gamma} : a^{-1}(U \cap B) \to \mathbb{C}$ *by*

$$\widetilde{\gamma} := \gamma \circ b.$$

Then

$$(\prod_{i=1}^r (x_i^*)^{d_i})\widetilde{\gamma}$$

analytically continues to an open neighborhood of $\{0\} \times (\relint(F) + i\lin(F))$*.*

Proposition 8. *(→ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $\mathfrak{c} \in \Omega(\mathcal{P})$ be a chamber such that $\lambda(\mathfrak{c}, \omega) < \infty$. Let $FL(\mathfrak{c}, \omega)$ be the flag as in Definition 18 and let $u_1, \ldots, u_r$ be the basis of $\operatorname{lin}(F(\mathfrak{c}, \omega))^\perp$ determined by $FL(\mathfrak{c}, \omega)$ in Definition 19. Let $u_1^*, \ldots, u_r^*$ be the dual basis of $u_1, \ldots, u_r$. For each $i \in [1, r]$, let $\mathfrak{P}(F_i(\mathfrak{c}, \omega), \mathfrak{c}, \omega) \in \mathbb{R}$ be as in Definition 16. Then, for every $i \in [1, r]$, $\mathfrak{P}(F_i(\mathfrak{c}, \omega), \mathfrak{c}, \omega) < 0$, and for all $u \in \operatorname{lin}(F(\mathfrak{c}, \omega))^\perp$ and $y \in F(\mathfrak{c}, \omega)_{\mathbb{C}}$,*

$$\omega(u + y) = \sum_{i=1}^{r} \mathfrak{P}(F_i(\mathfrak{c}, \omega), \mathfrak{c}, \omega) u_i^*(u) + \lambda(\mathfrak{c}, \omega).$$

To apply Proposition 7 to γ as in Definition 6, we need to construct a suitable open neighborhood of $\operatorname{relint}(F) + i \operatorname{lin}(F)$ in $\mathbb{C}^k$. We will use U_F from Lemma 6.

Lemma 6. *(→ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let F be a face of $\mathcal{P}$ of positive codimension. Define*

$$\widetilde{\mathcal{P}} := \bigcup_{q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})} Z(q).$$

For $x \in \operatorname{relint}(F)$, let $V_x \in \Omega(\widetilde{\mathcal{P}})$ be the chamber of $\widetilde{\mathcal{P}}$ containing x. Then V_x is independent of x; denote this common chamber by V_F. It is localized at F in the following sense:

1. $\operatorname{relint}(F) \subsetneq V_F$.
2. *$V_F \cap Z(q) = \emptyset$ for every $q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})$.*
3. *For every $q \in \mathfrak{Q}(F, \mathcal{P})$, $V_F \cap Z(q)$ is nonempty and connected.*

For $U_F := V_F + i\mathbb{R}^k$, the corresponding complex open neighborhood has the analogous properties:

1. $\operatorname{relint}(F) + i \operatorname{lin}(F) \subsetneq U_F$.
2. *$U_F \cap Z_{\mathbb{C}}(q) = \emptyset$ for every $q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})$.*
3. *For every $q \in \mathfrak{Q}(F, \mathcal{P})$, $U_F \cap Z_{\mathbb{C}}(q)$ is nonempty and connected.*

Lemma 7. *(→ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\mathfrak{c} \in \Omega(\mathcal{P})$, and let $F \subseteq \bar{\mathfrak{c}}$ be a face of positive codimension. Let V_F be the localized real open neighborhood from Lemma 6. Then $\mathfrak{c} \subseteq V_F$. In particular, if $\mathcal{P}$ has at least one positive codimension face, then every chamber is contained in V_F for some positive codimension face F in its closure.*

Lemma 8. *(→ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement, and let F be a face of $\mathcal{P}$ of positive codimension. Let $U \subseteq \mathbb{C}^k$ be open such that $\operatorname{relint}(F) + i \operatorname{lin}(F) \subsetneq U$ and $U \cap Z_{\mathbb{C}}(q) = \emptyset$ for every $q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})$. Assume further that $U \cap Z_{\mathbb{C}}(q)$ has finitely many connected components for every $q \in \mathfrak{Q}(F, \mathcal{P})$. Then $\widehat{R}(U, \mathcal{P}) = R(U, \mathcal{P})$.*

We can now define the polynomial $p_{\mathfrak{c},\omega}$.

Definition 30 ($p_{\mathfrak{c},\omega}$)**.** *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$. Let $\mathfrak{c} \in \Omega(\mathcal{P})$ be a chamber such that $\lambda(\mathfrak{c}, \omega) < \infty$. Set $F := F(\mathfrak{c}, \omega)$ and $r := \operatorname{codim}(F)$. Let $FL(\mathfrak{c}, \omega)$ be the flag from Definition 18, and let $u_1, \ldots, u_r$ be the basis of $\operatorname{lin}(F)^\perp$*

determined by $FL(\mathfrak{c},\omega)$ *in Definition 19. Let* $u_1^*,\dots,u_r^*$ *be the dual basis. Let* $\eta_H := u_1^*\wedge\cdots\wedge u_r^*$ *and let* $\lambda_F\in\Lambda^{k-r}\operatorname{lin}(F)^*_{\mathbb{C}}$ *be the unique form such that* $\Pi_{\mathbb{C}^k}=\eta_H\wedge\lambda_F$*, where* $\Pi_{\mathbb{C}^k}$ *is the standard holomorphic volume form on* $\mathbb{C}^k$*. Let* $x_1,\dots,x_r$ *be the standard coordinates for* $\mathbb{C}^r$*, and let* $x_1^*,\dots,x_r^*$ *be the dual basis. Let* $\gamma\in\widehat{R}(\mathbb{C}^k,\mathcal{P})$ *be as in Definition 6. Let* V_F *be the localized real open neighborhood from Lemma 6, and set* $U:=V_F+i\mathbb{R}^k$*, so that the setting of Definition 39 applies and* $\gamma\in R(U,\mathcal{P})$*. Let* $d_1,\dots,d_r\in\mathbb{Z}_{\geq 0}$ *and* $D_1,\dots,D_r\in\mathbb{Z}_{\geq 1}$ *be as in Definition 29, for* F *and* $FL(\mathfrak{c},\omega)$*, and let* $\widetilde{\gamma}$ *be as in Proposition 7. For each* $i\in[1,r]$*, let* $\mathfrak{P}(F_i(\mathfrak{c},\omega),\mathfrak{c},\omega)$ *be as in Definition 16. By Proposition 8,* $\mathfrak{P}(F_i(\mathfrak{c},\omega),\mathfrak{c},\omega)<0$ *for all* $i\in[1,r]$*.*

Define $\Upsilon:=\{\beta\in\mathbb{Z}^r_{\geq 0}:\beta_1\leq\cdots\leq\beta_r,\forall i\in[1,r]\ \beta_i\leq D_i-i\}$*. Define the polynomial* $p_{\mathfrak{c},\omega}:\mathbb{R}\to\mathbb{R}$ *by*

$$\begin{aligned}&p_{\mathfrak{c},\omega}(t)\\&:=\sum_{\beta\in\Upsilon}t^{\beta_r}\prod_{i=1}^{r}\frac{(-\mathfrak{P}(F_i(\mathfrak{c},\omega),\mathfrak{c},\omega))^{\beta_i-\beta_{i-1}}}{(\beta_i-\beta_{i-1})!(D_i-i-\beta_i)!}\\&\qquad\cdot\frac{\int_{c+i\operatorname{lin}(F)}\left[(\prod_{i=1}^{r}\partial_{x_i}^{D_i-i-\beta_i})(\prod_{i=1}^{r}(x_i^*)^{D_i}\widetilde{\gamma})\right]\big|_{x=0}\lambda_F}{(2\pi i)^{k-r}},\end{aligned}$$

where c *is any point in* $\operatorname{relint}(F)$ *and with the convention* $\beta_0=0$*.*

Corollary 2. *(→ proof)* $p_{\mathfrak{c},\omega}$ *is independent of* $c\in\operatorname{relint}(F(\mathfrak{c},\omega))$*.* $p_{\mathfrak{c},\omega}$ *has real coefficients.*

This completes our construction of every object that appears in the second version of the main theorem, Theorem 3.

3 Hironaka's theorem and integration

Using Hironaka's theorem, we will present a convenient way to view W through charts on the blow-up in which the analytic factors in the main setting become monomials, up to analytic units. The original functions f_i then become monomials times positive smooth units. We also record a mild generalization in which one simultaneously resolves an arbitrary finite family of additional analytic functions. This is summarized by Theorem 7. Then, by Theorem 8, this will help us compute integrals over W, and in particular, study the zeta function.

3.1 Resolution of singularities

Theorem 5 (Hironaka's theorem; compare with [16, 35])**.** *Let $x \in \mathbb{R}^d$. Let f be analytic on an open set in $\mathbb{R}^d$ containing x. Assume f is not identically zero on any open neighborhood of x. Then there exists a triple (W, U, g) where*

1. *$W \subseteq \mathbb{R}^d$ is an open neighborhood of x.*
2. *U is a real analytic manifold.*
3. *$g : U \to W$ is a proper real analytic map.*

such that

1. *For $W_0 = \{w \in W : f(w) = 0\}$ and $U_0 = \{u \in U : f \circ g(u) = 0\}$, $g|_{U \setminus U_0} : U \setminus U_0 \to W \setminus W_0$ is an analytic isomorphism.*
2. *For any $y \in U$, there exists a chart (U_y, π), where U_y is an open neighborhood of y and $\pi : U_y \to \pi(U_y) \subseteq \mathbb{R}^d$ is an analytic isomorphism, such that $\pi(y) = 0$ and there exist integers $k_1, \ldots, k_d, h_1, \ldots, h_d \in \mathbb{Z}_{\geq 0}$ and analytic units $a, b : \pi(U_y) \to \mathbb{R}^\times$ such that*

$$f \circ g \circ \pi^{-1} = a \prod_{l=1}^{d} u_l^{k_l}$$
$$\det D[g \circ \pi^{-1}] = b \prod_{l=1}^{d} u_l^{h_l},$$

where $D[\cdot]$ denotes the Jacobian of the map $\cdot$.

Remark 6. *Watanabe's presentation in [35, Theorem 2.3] differs slightly. First, Watanabe assumes $x = 0$, which is unimportant. Second, Watanabe requires that $f(x) = 0$. This forces $k_l > 0$ for some l, which allows one to package the unit $a : \pi(U_y) \to \mathbb{R}^\times$ into u_l, so that in the new coordinates, $f = S \prod_{l=1}^{d} u_l^{k_l}$ for $S = \pm 1$. In our setting, we allow $a : \pi(U_y) \to \mathbb{R}^\times$ to be non-constant, so that in the case $f(x) \neq 0$, we can set g as the identity map on an open neighborhood of x where f is a unit, set $a = f$, and set $0 = k_1 = \cdots = k_d = h_1 = \cdots = h_d$.*

Remark 7. *When $k = 1$ in the zeta function, one can avoid certain complications in studying the zeta function by making the unit, a, constant. In the case $k = 1$, at a certain stage our work will*

reduce to studying integrals of the form

$$\int_{[0,1]^d} \prod_{l=1}^{d} u_l^{k_l z + h_l} a(u)^z \omega(u) du,$$

where ω is smooth on an open neighborhood of $[0,1]^d$ and z is a complex number. If we had $a = 1$, this would be

$$\int_{[0,1]^d} \prod_{l=1}^{d} u_l^{k_l z + h_l} \omega(u) du,$$

which allows z to only interact with the monomial part. But for our case, when $k > 1$, we cannot make each f_i simultaneously monomial up to a constant, so we will have to handle factors of the form $a_i(u)^{z_i}$ anyway.

Lemma 9. *($\to$ proof) Let f be analytic on an open set in $\mathbb{R}^d$. Assume f is not identically zero around x.*

1. *There exists a triple (W, U, g) as in Theorem 5 such that W is connected.*
2. *If W is connected, g is surjective.*

Proof of Lemma 9. Let W' be the connected component of W that contains x. Set $U' := g^{-1}(W')$, which is an open subset of U, hence a real analytic manifold. Define $g' : U' \to W'$ by $g' := g|_{U'}$. We check that g' is proper.

Let $K \subseteq W'$ be compact. Then $g'^{-1}(K) = g^{-1}(K) \cap U'$ because $g' = g|_{U'}$. Since $g^{-1}(K) \subseteq U'$ by definition of U', we have $g'^{-1}(K) = g^{-1}(K)$. Because g is proper, $g^{-1}(K)$ is compact in the topology of U. In any topological space, if $A \subseteq B \subseteq C$, A is compact in the topology of C, and B is open, then A is compact in the topology of B. Thus $g'^{-1}(K)$ is compact in the topology of U', so g' is proper. It is clear that (W', U', g') satisfies the remaining conclusions of Theorem 5.

Now suppose W is connected. By [19, Theorem A.57], proper maps to locally compact Hausdorff spaces are closed. So g is closed. Since U is closed as a subset of itself, $g(U)$ is closed in the topology of W. Because $g|_{U \setminus U_0} : U \setminus U_0 \to W \setminus W_0$ is an analytic isomorphism, $g(U) \supseteq g(U \setminus U_0) = W \setminus W_0$. W is a connected open neighborhood of x, f is not identically zero in an open neighborhood of x, and f is analytic. Consequently, $W \setminus W_0$ is dense in W. Then because $g(U) \supseteq W \setminus W_0$, $g(U)$ is dense in W. But $g(U)$ is already closed, so $g(U) = W$. □

Theorem 6 (Simultaneous resolution of singularities)**.** *($\to$ proof) Let k be an integer and $f_1, \ldots, f_k$ be real analytic functions on an open set in $\mathbb{R}^d$ which contains x. Assume that for all $1 \leq i \leq k$, f_i is not identically zero around x. Then there exists a triple (W, U, g),*

1. *$W \subseteq \mathbb{R}^d$ is a connected open neighborhood of x.*
2. *U is a real analytic manifold.*
3. *$g : U \to W$ is a surjective proper real analytic map.*

which satisfies the following conditions:

1. *g is an analytic isomorphism between $U \setminus U_0$ and $W \setminus W_0$, where $W_0 = \bigcup_i \{w \in W : f_i(w) = 0\}$ and $U_0 = \bigcup_i \{u \in U : f_i \circ g(u) = 0\}$.*

2. *For any $y \in U$, there exists a chart (U_y, π), where U_y is an open neighborhood of y and $\pi : U_y \to \pi(U_y) \subseteq \mathbb{R}^d$ is an analytic isomorphism, such that $\pi(y) = 0$ and for each $1 \leq i \leq k$ there exists an analytic unit $a_i : \pi(U_y) \to \mathbb{R}^\times$ and $k_{i1}, \ldots, k_{id} \in \mathbb{Z}_{\geq 0}$ such that*

$$f_i \circ g \circ \pi^{-1} = a_i \prod_{l=1}^{d} u_l^{k_{il}}.$$

Further, there exists an analytic unit $b : \pi(U_y) \to \mathbb{R}^\times$ and $h_1, \ldots, h_d \in \mathbb{Z}_{\geq 0}$ such that

$$\det D[g \circ \pi^{-1}] = b \prod_{l=1}^{d} u_l^{h_l}.$$

Proof of Theorem 6. Apply Theorem 5 to $f = \prod_{i=1}^{k} f_i$. By Lemma 9, we can choose W connected and g surjective. By Theorem 5,

$$\prod_{i=1}^{k} (f_i \circ g \circ \pi^{-1}) = f \circ g \circ \pi^{-1}(u) = a \prod_{l=1}^{d} u_l^{k_l}.$$

Work in the real analytic local ring $\mathcal{O}_{\mathbb{R}^d,0} \simeq \mathbb{R}\{u_1, \ldots, u_d\}$. This is a unique factorization domain and the germs of the coordinate functions u_l are prime. The germ of each $f_i \circ g \circ \pi^{-1}$ divides the germ of $a \prod_{l=1}^{d} u_l^{k_l}$, with a a unit. Hence, in $\mathcal{O}_{\mathbb{R}^d,0}$,

$$f_i \circ g \circ \pi^{-1} = a_i \prod_{l=1}^{d} u_l^{k_{il}}$$

for some integers $k_{i1}, \ldots, k_{id} \in \mathbb{Z}_{\geq 0}$ and some unit germ a_i. After replacing U_y by a smaller open neighborhood, the unit germ is represented by an analytic unit $a_i : \pi(U_y) \to \mathbb{R}^\times$ and the displayed identity holds on $\pi(U_y)$.

$\det D[g \circ \pi^{-1}] = b \prod_{l=1}^{d} u_l^{h_l}$ is inherited directly from Theorem 5. □

Remark 8. *Watanabe also proves a version of the simultaneous resolution of singularities in [35, Theorem 2.8]. However, the notation in Theorem 2.8 appears inconsistent with the surrounding definitions, so for clarity we give a self-contained statement and proof.*

3.2 Preparation for the standard form

To arrive at our standard form in Theorem 7, we need to perform various normalizations and construct partitions of unity. This subsection is long and important only to arrive at the standard form. Readers are advised to skip this subsection and the proof of the existence of the standard form, which depends on the lemmas in this subsection. Instead, the reader should understand that, morally speaking, the standard form exists as a consequence of applying Hironaka's theorem at each point of W_0, as in Definition 31, and then taking a finite sub-cover because W is assumed compact.

Definition 31. *Assume the main setting of Definition 1. Let $\mathcal{T}$ be a finite index set. For each $t \in \mathcal{T}$, let ϖ_t be analytic on X, and assume ϖ_t is not identically zero on any connected component of X.*

Define

$$\begin{aligned} X_0 &:= \{x \in X : \exists 1 \le i \le k,\ \widetilde{f}_i(x) = 0\}, \\ X_{\text{res}} &:= \{x \in X : \exists 1 \le i \le k,\ \widetilde{f}_i(x) = 0\} \\ &\quad \cup \{x \in X : \exists 1 \le s \le m,\ \pi_s(x) = 0\} \\ &\quad \cup \{x \in X : \exists t \in \mathcal{T},\ \varpi_t(x) = 0\}, \\ W_0 &:= X_0 \cap W, \\ W_{\text{res}} &:= X_{\text{res}} \cap W. \end{aligned}$$

W_0 is exactly the set $\{w \in W : \exists i \in [1,k],\ f_i(w) = 0\}$ by Remark 1. It is where charts are needed, because outside an open neighborhood of W_0, the functions $f_1, \ldots, f_k$ are uniformly bounded below on W. But we will apply the simultaneous resolution to the larger analytic family

$$\widetilde{f}_1, \ldots, \widetilde{f}_k, \pi_1, \ldots, \pi_m, \varpi_t \ (t \in \mathcal{T}),$$

so the resulting map is an analytic isomorphism only on the complement of X_{res}.

Also, the analytic functions ϖ_t are not needed to prove the main theorem. We include them so that future applications can use the standard form while resolving the singularities of finitely many additional analytic functions that do not appear in our setting.

Lemma 10. *(→ proof) Let X be a topological space. Let $W \subseteq X$ be closed such that $W = \overline{W^\circ}$. Let $x \in W$. Let $W_x \subseteq X$ be an open neighborhood of x, let U_x be a topological space and let*

$$g_x : U_x \to W_x$$

be a proper surjective continuous map. Define

$$K_x := g_x^{-1}(x) \cap \overline{g_x^{-1}(W^\circ \cap W_x)}.$$

Suppose x has an open neighborhood C_x in W_x, whose closure in W_x is compact. Then K_x is non-empty and compact.

In particular, if X is locally compact Hausdorff, then K_x is non-empty and compact.

Proof of Lemma 10. Since W° is dense in W, and W_x is an open neighborhood of x, we have

$$x \in \overline{W^\circ \cap W_x}.$$

Then there exists a net $x_i \in W^\circ \cap W_x$ such that

$$x_i \to x.$$

For each i, because g_x is surjective, there exists a point $u_i \in U_x$ such that

$$g_x(u_i) = x_i.$$

Since $x_i \to x$ and C_x is an open neighborhood of x, eventually we have $x_i \in C_x$. Then

$$u_i \in g_x^{-1}(C_x) \subseteq g_x^{-1}(\overline{C}_x).$$

Since g_x is proper and $\overline{C}_x$ is compact, $g_x^{-1}(\overline{C}_x)$ is compact. Hence, after passing to a subnet, we may assume

$$u_{i_j} \to u,$$

for some $u \in g_x^{-1}(\overline{C}_x)$.

By continuity of g_x,

$$g_x(u) = \lim_j g_x(u_{i_j}) = \lim_j x_{i_j} = x,$$

so $u \in g_x^{-1}(x)$. Also, each u_{i_j} lies in $g_x^{-1}(W^\circ \cap W_x)$, hence

$$u \in \overline{g_x^{-1}(W^\circ \cap W_x)}.$$

Therefore

$$u \in g_x^{-1}(x) \cap \overline{g_x^{-1}(W^\circ \cap W_x)} = K_x,$$

so K_x is non-empty. Finally, since g_x is proper and $\{x\}$ is compact, $g_x^{-1}(x)$ is compact. The set

$$\overline{g_x^{-1}(W^\circ \cap W_x)}$$

is closed in U_x, so K_x is closed in $g_x^{-1}(x)$. Hence K_x is compact. □

Lemma 11. *(→ proof) Assume the main setting of Definition 1 and the notation of Definition 31. Let $x \in W$. Let (W_x, U_x, g_x) be the triple given by Theorem 6 for the family*

$$\widetilde{f}_1, \ldots, \widetilde{f}_k, \pi_1, \ldots, \pi_m, \varpi_t \ (t \in \mathcal{T})$$

at x. Let K_x be as in Lemma 10. For any $y \in K_x$, there exists a chart $(U_{x,y}, \pi_{x,y})$, where $U_{x,y}$ is a connected open neighborhood of y and $\pi_{x,y} : U_{x,y} \to \widetilde{U}_{x,y} \subseteq \mathbb{R}^d$ is an analytic isomorphism, such that $\pi_{x,y}(y) = 0$, and there exist positive smooth units

$$a_{x,y,i} : \widetilde{U}_{x,y} \to \mathbb{R}_{>0},$$

together with positive analytic units

$$d_{x,y,s}, \widetilde{b}_{x,y} : \widetilde{U}_{x,y} \to \mathbb{R}_{>0},$$

together with, for each $t \in \mathcal{T}$, an analytic unit

$$c_{x,y,t} : \widetilde{U}_{x,y} \to \mathbb{R}^\times,$$

together with exponents

$$k_{x,y,i,l},\ p_{x,y,s,l},\ r_{x,y,l} \in \mathbb{Z}_{\geq 0},$$

and, for each $t \in \mathcal{T}$*, exponents*

$$n_{x,y,t,l} \in \mathbb{Z}_{\geq 0},$$

such that on $\widetilde{U}_{x,y}$*, with* $g_{x,y} : \widetilde{U}_{x,y} \to W_x$ *defined by* $g_{x,y} := g_x \circ \pi_{x,y}^{-1}$*,*

$$\begin{aligned} f_i \circ g_{x,y} &= a_{x,y,i} \prod_{l=1}^{d} u_l^{k_{x,y,i,l}}, \\ \pi_s \circ g_{x,y} &= d_{x,y,s} \prod_{l=1}^{d} u_l^{p_{x,y,s,l}}, \\ \varpi_t \circ g_{x,y} &= c_{x,y,t} \prod_{l=1}^{d} u_l^{n_{x,y,t,l}}, \\ |\det Dg_{x,y}| &= \widetilde{b}_{x,y} \prod_{l=1}^{d} |u_l|^{r_{x,y,l}}. \end{aligned}$$

Let X_{res} *be as in Definition 31. Then*

$$g_{x,y}|_{\widetilde{U}_{x,y} \setminus g_{x,y}^{-1}(X_{\mathrm{res}})} : \widetilde{U}_{x,y} \setminus g_{x,y}^{-1}(X_{\mathrm{res}}) \to g_x(U_{x,y}) \setminus X_{\mathrm{res}}$$

is an analytic isomorphism.

Proof of Lemma 11. First, restrict $U_{x,y}$ to a connected open neighborhood of y.

Next, Theorem 6 gives analytic units $\widetilde{a}_{x,y,i}, d_{x,y,s}, b_{x,y}$, analytic units $c_{x,y,t}$, and exponents with the formulas

$$\begin{aligned} \widetilde{f}_i \circ g_{x,y} &= \widetilde{a}_{x,y,i} \prod_{l=1}^{d} u_l^{k_{x,y,i,l}}, \\ \pi_s \circ g_{x,y} &= d_{x,y,s} \prod_{l=1}^{d} u_l^{p_{x,y,s,l}}, \\ \varpi_t \circ g_{x,y} &= c_{x,y,t} \prod_{l=1}^{d} u_l^{n_{x,y,t,l}}, \\ \det Dg_{x,y} &= b_{x,y} \prod_{l=1}^{d} u_l^{r_{x,y,l}} \end{aligned}$$

on $\widetilde{U}_{x,y}$.

Composing $\pi_{x,y}$ with a positive scalar dilation preserves the form of the formulas, so we may also assume

$$\overline{\Delta} \subsetneq \widetilde{U}_{x,y}.$$

Next, define

$$E_{x,y} := \pi_{x,y}\big(g_x^{-1}(W^\circ \cap W_x) \cap U_{x,y}\big).$$

Because $g_x^{-1}(W^\circ \cap W_x) \cap U_{x,y}$ is open in $U_{x,y}$, $E_{x,y}$ is open in $\widetilde{U}_{x,y}$. Since $y \in K_x$, we have

$$y \in \overline{g_x^{-1}(W^\circ \cap W_x)},$$

hence

$$0 = \pi_{x,y}(y) \in \overline{E_{x,y}}.$$

Because Δ° is an open neighborhood of 0 and $\Delta^\times$ is dense in Δ°, there exists

$$u^* \in E_{x,y} \cap \Delta^\times.$$

Now compose $\pi_{x,y}$ with the coordinate reflection

$$u \longmapsto \varepsilon u, \qquad \varepsilon_l := \operatorname{sign}(u_l^*) \in \{\pm 1\}.$$

This again preserves the displayed monomial form, changing only the units by signs. After this replacement, we have

$$u^* \in (0,1)^d.$$

Now $g_{x,y}(u^*) \in W^\circ \cap W_x$. Since $u_l^* > 0$ for all l, every monomial factor in the formulas is positive at u^*. Then because $f_1, \dots, f_k, \pi_1, \dots, \pi_m \geq 0$ on W, we have

$$f_i(g_{x,y}(u^*)) > 0 \qquad \pi_s(g_{x,y}(u^*)) > 0,$$

and thus

$$\big((\upsilon_i \circ g_{x,y})\widetilde{a}_{x,y,i}\big)(u^*) > 0 \qquad d_{x,y,s}(u^*) > 0.$$

Since $U_{x,y}$ is connected, $\widetilde{U}_{x,y}$ is connected. Each of the units is continuous and nowhere-vanishing on $\widetilde{U}_{x,y}$, hence has constant sign on $\widetilde{U}_{x,y}$. Since we found a point in $\widetilde{U}_{x,y}$ where $\big((\upsilon_i \circ g_{x,y})\widetilde{a}_{x,y,i}\big)$ and $d_{x,y,s}$ are positive, each of them is positive everywhere on $\widetilde{U}_{x,y}$. Define

$$a_{x,y,i} := (\upsilon_i \circ g_{x,y})\widetilde{a}_{x,y,i}.$$

Then $a_{x,y,i}$ is a positive smooth unit on $\widetilde{U}_{x,y}$, and

$$f_i \circ g_{x,y} = a_{x,y,i} \prod_{l=1}^{d} u_l^{k_{x,y,i,l}}.$$

Also, Theorem 6 provides $\det Dg_{x,y} = b_{x,y} \prod_{l=1}^{d} u_l^{r_{x,y,l}}$, which gives

$$|\det Dg_{x,y}| = |b_{x,y}| \prod_{l=1}^{d} |u_l|^{r_{x,y,l}}.$$

Since $b_{x,y}$ takes values in $\mathbb{R}^\times$, $|b_{x,y}|$ is smooth. Re-labelling $b_{x,y} \leftarrow |b_{x,y}|$, then we have $|\det Dg_{x,y}| = \widetilde{b}_{x,y} \prod_{l=1}^{d} |u_l|^{r_{x,y,l}}$, where $\widetilde{b}_{x,y} := |b_{x,y}|$ is a positive analytic unit.

Since $g_x(y) = x$ and $\pi_{x,y}(y) = 0$, we have

$$g_{x,y}(0) = g_x(\pi_{x,y}^{-1}(0)) = g_x(y) = x.$$

Because Theorem 6 was applied to the family

$$\widetilde{f}_1, \ldots, \widetilde{f}_k, \pi_1, \ldots, \pi_m, \varpi_t \ (t \in \mathcal{T}),$$

the restriction

$$g_x\big|_{U_x \setminus g_x^{-1}(X_{\text{res}})} : U_x \setminus g_x^{-1}(X_{\text{res}}) \to W_x \setminus X_{\text{res}}$$

is an analytic isomorphism. Composing with the analytic isomorphism $\pi_{x,y}^{-1}$,

$$g_{x,y}\big|_{\pi_{x,y}(U_{x,y} \setminus g_x^{-1}(X_{\text{res}}))} : \pi_{x,y}(U_{x,y} \setminus g_x^{-1}(X_{\text{res}})) \to g_x(U_{x,y}) \setminus X_{\text{res}}$$

is an analytic isomorphism. Because $\pi_{x,y}$ is an analytic isomorphism,

$$\pi_{x,y}(U_{x,y} \setminus g_x^{-1}(X_{\text{res}})) = \widetilde{U}_{x,y} \setminus \pi_{x,y}(U_{x,y} \cap g_x^{-1}(X_{\text{res}})) = \widetilde{U}_{x,y} \setminus g_{x,y}^{-1}(X_{\text{res}}),$$

thus

$$g_{x,y}\big|_{\widetilde{U}_{x,y} \setminus g_{x,y}^{-1}(X_{\text{res}})} : \widetilde{U}_{x,y} \setminus g_{x,y}^{-1}(X_{\text{res}}) \to g_x(U_{x,y}) \setminus X_{\text{res}}$$

is an analytic isomorphism. □

In Lemma 12, we only need the assumption that g_x is closed, not that g_x is proper. By [19, Theorem A.57], proper maps to locally compact Hausdorff spaces are closed, so this is a weaker assumption in that case.

Lemma 12. *(→ proof) Let X be a topological space. Let $x \in X$. Let $W_x \subseteq X$ be an open neighborhood of x, let U_x be a topological space, and let*

$$g_x : U_x \to W_x$$

be a surjective closed continuous map. Let $W \subseteq X$ be closed such that $x \in W$ and $W = \overline{W^\circ}$. Let $S_x \subseteq U_x$ be open. Let K_x be as in Lemma 10 and suppose that $K_x \subseteq S_x$. Then there exists an open set Q_x such that

$$x \in Q_x \subseteq W_x \qquad Q_x \cap W^\circ \subseteq g_x(S_x)$$

Proof of Lemma 12. Let $R_x := (U_x \setminus S_x) \cap \overline{g_x^{-1}(W^\circ \cap W_x)}$. R_x is closed. So because g_x is closed, $g_x(R_x) \subseteq W_x$ is closed in the topology of W_x. Let $Q_x := W_x \setminus g_x(R_x)$. So Q_x is open in the topology of W_x. Since W_x is open in X, Q_x is open in X.

Let $w \in Q_x \cap W^\circ$. Then $w \in W_x = g_x(U_x)$ since g_x is surjective. So there exists $u \in U_x$ such that $g_x(u) = w$. Since $w \notin g_x(R_x)$, $u \notin R_x$, so $u \in S_x$ or $u \notin \overline{g_x^{-1}(W^\circ \cap W_x)}$. In the second case, $w = g_x(u) \notin W^\circ \cap W_x$, which is a contradiction. Thus $u \in S_x$. So $w = g_x(u) \in g_x(S_x)$. Thus $Q_x \cap W^\circ \subseteq g_x(S_x)$.

Suppose $x \notin Q_x$. Then $x \in g_x(R_x)$. Then there exists $u \in R_x$ such that $g_x(u) = x$. Then $u \in \overline{g_x^{-1}(W^\circ \cap W_x)}$ and $u \in g_x^{-1}(x)$, so $u \in K_x$. Since $K_x \subseteq S_x$, $u \in S_x$. But since $u \in R_x$, $u \notin S_x$. So we have a contradiction. So $x \in Q_x$. □

Lemma 13. *(→ proof) Let X be a smooth d-dimensional manifold. Let $W \subseteq X$ be closed such that $W = \overline{W^\circ}$. Let $X_{\text{res}} \subseteq X$ be any closed set and let $A \subseteq W$ be any subset. For each $x \in A$,*

1. *Let $W_x \subseteq X$ be open, let U_x be a smooth d-dimensional manifold, and let*

$$g_x : U_x \to W_x$$

be a surjective proper continuous map. Set

$$E_x := g_x^{-1}(X_{\mathrm{res}}).$$

Assume that

$$g_x|_{U_x \setminus E_x} : U_x \setminus E_x \to W_x \setminus X_{\mathrm{res}}$$

is a diffeomorphism.

2. *Let*

$$B_x \subseteq g_x^{-1}(x)$$

be any subset.

3. *For each $y \in B_x$, let*

$$\pi_{x,y} : U_{x,y} \to \widetilde{U}_{x,y} \subseteq \mathbb{R}^d$$

be a smooth chart on U_x, where $U_{x,y}$ is an open neighborhood of y and $\pi_{x,y}(y) = 0$.

4. *Let*

$$\chi_{x,y} \subseteq \widetilde{U}_{x,y}$$

be an open neighborhood of 0. Set

$$\chi'_{x,y} := \pi_{x,y}^{-1}(\chi_{x,y}) \qquad \chi''_{x,y} := g_x(\chi'_{x,y})$$

5. *Let K_x be as in Lemma 10. Suppose $K_x \subseteq \bigcup_{y \in B_x} \chi'_{x,y}$.*

6. *Let $\{\vartheta_{x,y}\}_{y \in B_x} \subseteq C^\infty(U_x, [0,1])$ be such that*

   (a) *$\{\vartheta_{x,y}\}_{y \in B_x}$ is locally finite.*

   (b) *$\operatorname{supp} \vartheta_{x,y} \subseteq \chi'_{x,y}$.*

   (c) *$\sum_{y \in B_x} \vartheta_{x,y} = 1$ on an open neighborhood $\widetilde{K}_x$ such that $K_x \subseteq \widetilde{K}_x \subseteq \bigcup_{y \in B_x} \chi'_{x,y}$.*

   (d) *$\sum_{y \in B_x} \vartheta_{x,y} \in [0,1]$ on U_x.*

7. *Let $Q_x \subseteq W_x$ be open such that $x \in Q_x$ and $Q_x \cap W^\circ \subseteq g_x(\widetilde{K}_x)$.*

8. *Let $\{\eta_x\}_{x \in A} \subseteq C^\infty(X, [0,1])$ be such that*

   (a) *$\{\eta_x\}_{x \in A}$ is locally finite.*

   (b) *$\operatorname{supp} \eta_x \subseteq Q_x$.*

   (c) *$\sum_{x \in A} \eta_x = 1$ on an open neighborhood $\widetilde{K}$.*

   (d) *$\sum_{x \in A} \eta_x \in [0,1]$ on X.*

Define $\psi_{x,y} : X \setminus X_{\text{res}} \to [0,1]$ *and* $\rho_{x,y} : \widetilde{U}_{x,y} \to [0,1]$ *by*

$$\psi_{x,y}(w) := \begin{cases} \eta_x(w)\vartheta_{x,y}((g_x|_{U_x \setminus E_x})^{-1}(w)), & : w \in W_x, \\ 0 & : w \notin W_x, \end{cases}$$
$$\rho_{x,y}(u) := \eta_x(g_x(\pi_{x,y}^{-1}(u)))\vartheta_{x,y}(\pi_{x,y}^{-1}(u))$$

Set

$$\Gamma_{x,y} := \pi_{x,y}(E_x \cap U_{x,y})$$

Then:

1. $\{\psi_{x,y}\}_{x \in A, y \in B_x}$ *is locally finite.*
2. $\psi_{x,y} \in C^\infty(X \setminus X_{\text{res}}, [0,1])$.
3. $\operatorname{supp} \psi_{x,y} \subseteq \chi''_{x,y} \setminus X_{\text{res}}$, *where* supp *is taken in the topology of* $X \setminus X_{\text{res}}$. *In particular,* $\operatorname{supp} \psi_{x,y} \subseteq W_x \setminus X_{\text{res}}$.
4. *Let* $\widetilde{W}_0 := \widetilde{K} \cap \bigcup_{x \in A} Q_x$ *and* $W_{\text{res}} := X_{\text{res}} \cap W$. *Then on* $\widetilde{W}_0 \cap W \setminus W_{\text{res}}$, $\sum_{x \in A} \sum_{y \in B_x} \psi_{x,y} = 1$.
5. *On* $X \setminus X_{\text{res}}$, $\sum_{x \in A} \sum_{y \in B_x} \psi_{x,y} \in [0,1]$.
6. *On* $\widetilde{U}_{x,y} \setminus \Gamma_{x,y}$, $\rho_{x,y} = \psi_{x,y} \circ g_x \circ \pi_{x,y}^{-1}$.
7. $\rho_{x,y} \in C^\infty(\widetilde{U}_{x,y}, [0,1])$.

If, in addition, for some $x \in A$ *and* $y \in B_x$, $\eta_x = 1$ *on an open neighborhood of* x *and* $\vartheta_{x,y} = 1$ *on an open neighborhood of* y, *then* $\rho_{x,y} = 1$ *on an open neighborhood of* 0.

Proof of Lemma 13. Write

$$h_x := (g_x|_{U_x \setminus E_x})^{-1} : W_x \setminus X_{\text{res}} \to U_x \setminus E_x.$$

This is an analytic isomorphism by hypothesis, thus smooth. Hence, on $W_x \setminus X_{\text{res}}$,

$$w \longmapsto \eta_x(w)\vartheta_{x,y}(h_x(w))$$

is smooth. Since $\operatorname{supp} \eta_x \subseteq Q_x \subseteq W_x$, η_x vanishes on an open neighborhood of $(X \setminus X_{\text{res}}) \setminus W_x$. So for all $w \in (X \setminus X_{\text{res}}) \setminus W_x$, there is an open neighborhood of w on which $\psi_{x,y} = 0$, so $\psi_{x,y}$ is smooth on an open neighborhood of w. So $\psi_{x,y}$ is smooth on $X \setminus X_{\text{res}}$.

On $W_x \setminus X_{\text{res}}$, $\psi_{x,y}$ is a product of functions taking values in $[0,1]$, so $\psi_{x,y}$ takes values in $[0,1]$. On $X \setminus W_x$, $\psi_{x,y} = 0$. So in either case, $\psi_{x,y}$ takes values in $[0,1]$. This proves (2).

If $\psi_{x,y}(w) \neq 0$, then $w \in W_x$, $\eta_x(w) \neq 0$, and $\vartheta_{x,y}(h_x(w)) \neq 0$. Thus

$$h_x(w) \in \operatorname{supp} \vartheta_{x,y} \subseteq \chi'_{x,y}.$$

Applying g_x gives

$$w = g_x(h_x(w)) \in g_x(\chi'_{x,y}) = \chi''_{x,y}.$$

Since $\psi_{x,y}$ is defined on $X \setminus X_{\text{res}}$, this yields

$$\operatorname{supp} \psi_{x,y} \subseteq \chi''_{x,y} \setminus X_{\text{res}},$$

which proves (3).

Let $w \in X \setminus X_{\text{res}}$. Because $\{\eta_x\}_{x\in A}$ is locally finite, there is an open neighborhood $O \subseteq X \setminus X_{\text{res}}$ containing w such that

$$\Omega := \{x \in A : \eta_x \text{ is non-zero at some point of } O\}$$

is finite. Let

$$\Omega^+ := \{x \in \Omega : w \in \operatorname{supp} \eta_x\}.$$

If $x \in \Omega \setminus \Omega^+$, choose an open neighborhood O_x^η of w on which $\eta_x = 0$. If $x \in \Omega^+$, then $w \in \operatorname{supp} \eta_x \subseteq Q_x \subseteq W_x$, so $u_x := h_x(w)$ is well-defined. Since $w \notin X_{\text{res}}$, $u_x \notin E_x$. By local finiteness of $\{\vartheta_{x,y}\}_{y\in B_x}$, there is an open neighborhood $O_x \subseteq U_x \setminus E_x$ of u_x such that

$$\Omega_x := \{y \in B_x : \vartheta_{x,y} \text{ is non-zero at some point of } O_x\}$$

is finite.

$$\widetilde{O} := O \cap \bigcap_{x\in\Omega^+} g_x(O_x) \cap \bigcap_{x\in\Omega\setminus\Omega^+} O_x^\eta.$$

Since Ω is finite, $\widetilde{O}$ is an open neighborhood of w in $X \setminus X_{\text{res}}$. If $\psi_{x,y}$ is non-zero at some point $t \in \widetilde{O}$, then $\eta_x(t) \neq 0$, so $x \in \Omega$. The construction of O_x^η rules out $x \in \Omega \setminus \Omega^+$, hence $x \in \Omega^+$. Then $t \in g_x(O_x)$, so $h_x(t) \in O_x$, and therefore $y \in \Omega_x$. Thus

$$\{(x, y) : \psi_{x,y} \text{ is non-zero at some point of } \widetilde{O}\} \subseteq \{(x, y) : x \in \Omega^+,\ y \in \Omega_x\},$$

which is finite. So $\{\psi_{x,y}\}_{x\in A, y\in B_x}$ is locally finite. This proves (1).

Let $w \in \widetilde{W}_0 \cap W^\circ \setminus X_{\text{res}}$ be arbitrary. Define $O := \{x \in A : w \in Q_x\}$. Since $w \in \widetilde{W}_0 \subseteq \bigcup_{x\in A} Q_x$, O is not empty.

Let $x \in O$. By hypothesis, $Q_x \cap W^\circ \subseteq g_x(\widetilde{K}_x)$. Thus $w \in g_x(\widetilde{K}_x)$. Thus $w \in W_x \setminus X_{\text{res}}$. Since $g_x|_{U_x\setminus E_x} : U_x \setminus E_x \to W_x \setminus X_{\text{res}}$ is a diffeomorphism, $u := (g_x|_{U_x\setminus E_x})^{-1}(w)$ is well-defined. Since $w \in g_x(\widetilde{K}_x)$, $u \in \widetilde{K}_x$. For each $y \in B_x$, since $w \in W_x$, by definition of $\psi_{x,y}$, $\psi_{x,y}(w) = \eta_x(w)\vartheta_{x,y}(u)$. Since $u \in \widetilde{K}_x$, $\sum_{y\in B_x} \psi_{x,y}(w) = \sum_{y\in B_x} \eta_x(w)\vartheta_{x,y}(u) = \eta_x(w)$. Thus, for all $x \in O$, $\sum_{y\in B_x} \psi_{x,y}(w) = \eta_x(w)$.

Let $x \in A \setminus O$. Then $w \notin Q_x$, so $w \notin \operatorname{supp} \eta_x$. Then by definition of $\psi_{x,y}$, for all $y \in B_x$, in either the case $w \in W_x$ or $w \notin W_x$, $\psi_{x,y}(w) = 0$. Thus $\sum_{x\in A} \sum_{y\in B_x} \psi_{x,y}(w) = \sum_{x\in O} \sum_{y\in B_x} \psi_{x,y}(w)$. Similarly, $\sum_{x\in A} \eta_x(w) = \sum_{x\in O} \eta_x(w)$.

Combining the facts that $\sum_{y\in B_x} \psi_{x,y}(w) = \eta_x(w)$ for all $x \in O$ and $\sum_{x\in A} \sum_{y\in B_x} \psi_{x,y}(w) = \sum_{x\in O} \sum_{y\in B_x} \psi_{x,y}(w)$,

$$\sum_{x\in A} \sum_{y\in B_x} \psi_{x,y}(w) = \sum_{x\in O} \sum_{y\in B_x} \psi_{x,y}(w) = \sum_{x\in O} \eta_x(w) = \sum_{x\in A} \eta_x(w) = 1.$$

This shows that $\sum_{x\in A}\sum_{y\in B_x}\psi_{x,y}=1$ on $\widetilde{W}_0\cap W^\circ\setminus X_{\rm res}$. Then by continuity and the fact that $W=\overline{W^\circ}$, $\sum_{x\in A}\sum_{y\in B_x}\psi_{x,y}=1$ on $\widetilde{W}_0\cap W\setminus X_{\rm res}$. Then by definition of $W_{\rm res}$, $W_{\rm res}=W\cap X_{\rm res}$. So $\widetilde{W}_0\cap W\setminus X_{\rm res}=\widetilde{W}_0\cap W\setminus W_{\rm res}$. This proves (4).

Now let $w\in X\setminus X_{\rm res}$ be arbitrary and fix $x\in A$. If $w\in W_x\setminus X_{\rm res}$, then $\psi_{x,y}$ is a product of functions taking values in $[0,1]$, so $\psi_{x,y}$ takes values in $[0,1]$. Let $u:=h_x(w)$. Then

$$\sum_{y\in B_x}\psi_{x,y}(w)=\eta_x(w)\sum_{y\in B_x}\vartheta_{x,y}(u)\le\eta_x(w).$$

If $w\notin W_x$, then $\sum_{y\in B_x}\psi_{x,y}(w)=0\le\eta_x(w)$.

Thus

$$\sum_{x\in A}\sum_{y\in B_x}\psi_{x,y}(w)\le\sum_{x\in A}\eta_x(w)\le 1.$$

This proves (5).

If $u\in\widetilde{U}_{x,y}\setminus\Gamma_{x,y}$, then $\pi_{x,y}^{-1}(u)\in U_{x,y}\setminus E_x\subseteq U_x\setminus E_x$, so $h_x(g_x(\pi_{x,y}^{-1}(u)))=\pi_{x,y}^{-1}(u)$. Hence

$$\begin{aligned}\psi_{x,y}(g_x(\pi_{x,y}^{-1}(u)))&=\eta_x(g_x(\pi_{x,y}^{-1}(u)))\vartheta_{x,y}(h_x(g_x(\pi_{x,y}^{-1}(u))))\\&=\eta_x(g_x(\pi_{x,y}^{-1}(u)))\vartheta_{x,y}(\pi_{x,y}^{-1}(u))\\&=\rho_{x,y}(u),\end{aligned}$$

which proves (6).

Finally, $\rho_{x,y}$ is a product of functions in $C^\infty(\widetilde{U}_{x,y},[0,1])$, so $\rho_{x,y}\in C^\infty(\widetilde{U}_{x,y},[0,1])$, which proves (7).

If $\eta_x=1$ on an open neighborhood of x and $\vartheta_{x,y}=1$ on an open neighborhood of y, then using

$$g_x(\pi_{x,y}^{-1}(0))=x,\qquad \pi_{x,y}^{-1}(0)=y,$$

continuity implies $\rho_{x,y}=1$ on an open neighborhood of 0. □

Lemma 14. *(→ proof) Let X be a regular Hausdorff space. Let $A\subseteq X$ be finite. Let $\{U_x\}_{x\in A}$ be open sets in X with $x\in U_x$ for every $x\in A$. Then there exist open sets $K_x,V_x\subseteq X$ such that*

1. *$x\in K_x\subseteq\overline{K_x}\subseteq V_x\subseteq U_x$.*
2. *For all $x\ne x'\in A$, $\overline{K_x}\cap\overline{V_{x'}}=\emptyset$.*
3. *$\bigcup_{x\in A}V_x=\bigcup_{x\in A}U_x$.*

Proof of Lemma 14. Because X is Hausdorff, and A is finite, there exist neighborhoods R_x of x such that the family $\{R_x\}_{x\in A}$ is pairwise disjoint: if $x\ne x'$, then $R_x\cap R_{x'}=\emptyset$.

Now $R_x\cap U_x$ is an open neighborhood of x. Because X is regular, there exist open sets O_x and K_x such that

$$x\in K_x\subseteq\overline{K_x}\subseteq O_x\subseteq\overline{O_x}\subseteq R_x\cap U_x.$$

In particular, the sets $\overline{O_x}$ are pairwise disjoint and $\overline{O_x} \subseteq U_x$. Define

$$V_x := U_x \setminus \bigcup_{x' \neq x} \overline{O_{x'}}.$$

Then each V_x is open. Because the closures $\overline{O_x}$ are pairwise disjoint, we have

$$\overline{O_x} \subseteq V_x.$$

Hence

$$x \in K_x \subseteq \overline{K_x} \subseteq O_x \subseteq \overline{O_x} \subseteq V_x \subseteq U_x,$$

so (1) holds.

Now let $x \neq x' \in A$. Since $\overline{O_x}$ is one of the closed sets removed in the definition of $V_{x'}$, we have

$$V_{x'} \subseteq X \setminus \overline{O_x} \subseteq X \setminus O_x.$$

As $X \setminus O_x$ is closed,

$$\overline{V_{x'}} \subseteq X \setminus O_x.$$

But $\overline{K_x} \subseteq O_x$, so

$$\overline{K_x} \cap \overline{V_{x'}} = \emptyset.$$

Thus (2) holds.

Finally, let $w \in \bigcup_{x \in A} U_x$. First suppose $w \notin \bigcup_{x \in A} \overline{O_x}$. There exists $x \in A$ with $w \in U_x$. Then

$$w \in U_x \setminus \bigcup_{x' \neq x} \overline{O_{x'}} = V_x.$$

If instead $w \in \bigcup_{x \in A} \overline{O_x}$, then because $\{\overline{O_x}\}_{x \in A}$ are pairwise disjoint, there is a unique $x_0 \in A$ such that $w \in \overline{O_{x_0}}$. Since $\overline{O_{x_0}} \subseteq U_{x_0}$ and $w \notin \overline{O_x}$ for $x \neq x_0$, we get

$$w \in U_{x_0} \setminus \bigcup_{x \neq x_0} \overline{O_x} = V_{x_0}.$$

So $\bigcup_{x \in A} U_x \subseteq \bigcup_{x \in A} V_x$. The reverse inclusion is immediate from $V_x \subseteq U_x$. Therefore

$$\bigcup_{x \in A} V_x = \bigcup_{x \in A} U_x.$$

This proves (3). □

Lemma 15. *(→ proof) Let M be a smooth manifold. Let $A \subseteq M$ be finite. Let $\{U_x\}_{x \in A}$ be an open cover of M such that $x \in U_x$. Then there exists, for each $x \in A$, $\rho_x \in C^\infty(M, [0,1])$ such that $\operatorname{supp} \rho_x \subseteq U_x$, $\sum_{x \in A} \rho_x = 1$ on M, and $\rho_x = 1$ on an open neighborhood of x.*

Proof of Lemma 15. By Lemma 14, there exists open V_x and K_x such that

1. $x \in K_x \subseteq \overline{K}_x \subseteq V_x \subseteq U_x$.

2. For all $x \neq x' \in A$, $\overline{K}_x \cap \overline{V}_{x'} = \emptyset$.

3. $\bigcup_{x\in A} V_x = \bigcup_{x\in A} U_x = M$.

$\{V_x\}_{x\in A}$ is an open cover of M. So there exists a partition of unity $\{\rho_x\}_{x\in A}$ on $\bigcup_{x\in A} V_x$ subordinate to $\{V_x\}_{x\in A}$. Because for all $x' \neq x$, $\operatorname{supp}\rho_{x'} \subseteq V_{x'}$, and $\overline{K}_x \cap V_{x'} = \emptyset$, only ρ_x has support on $\overline{K}_x$. So $\rho_x = 1$ on $\overline{K}_x \supseteq K_x \ni x$. □

Lemma 16. *(→ proof) Let M be a smooth manifold. Let $A \subseteq M$ be finite. Let $\{U_x\}_{x\in A}$ be a collection of open sets in M such that $x \in U_x$. Let $M_0 \subseteq M$ be closed such that $M_0 \subseteq \bigcup_{x\in A} U_x$. Then there exists an open set $\widetilde{M}_0$ such that $M_0 \subseteq \widetilde{M}_0 \subseteq M$, and for each $x \in A$, there exists $\rho_x \in C^\infty(M, [0,1])$ such that $\operatorname{supp}\rho_x \subseteq U_x$, $\sum_{x\in A}\rho_x = 1$ on $\widetilde{M}_0$, $\sum_{x\in A}\rho_x \in [0,1]$ on M, and $\rho_x = 1$ on an open neighborhood of x.*

Proof of Lemma 16. Suppose $\bigcup_{x\in A} U_x = M$. Then this statement is immediately equivalent to Lemma 15 with $\widetilde{M}_0 = M$.

Otherwise, there is $w \in M \setminus \bigcup_{x\in A} U_x$. Since M is a manifold, M is a normal Hausdorff space, so there exists open $\widetilde{M}_0$ such that $M_0 \subseteq \widetilde{M}_0 \subseteq \overline{\widetilde{M}}_0 \subseteq \bigcup_{x\in A} U_x$. Let $V := M \setminus \overline{\widetilde{M}}_0$. Then $M \setminus \bigcup_{x\in A} U_x \subseteq V$, so $\{U_x\}_{x\in A} \cup \{V\}$ is an open cover of M. Since $w \in M \setminus \bigcup_{x\in A} U_x \subseteq V$, we may apply Lemma 15 with the finite set $B = A \cup \{w\}$ and the open cover $\{U_x\}_{x\in A} \cup \{V\}$.

So there exists $\rho_w \in C^\infty(M, [0,1])$, and for each $x \in A$, $\rho_x \in C^\infty(M, [0,1])$, such that $\operatorname{supp}\rho_x \subseteq U_x$, $\operatorname{supp}\rho_w \subseteq V$, $\rho_w + \sum_{x\in A}\rho_x = 1$ on M, and $\rho_x = 1$ on an open neighborhood of x.

Since $\operatorname{supp}\rho_w \subseteq V$ and $V := M \setminus \overline{\widetilde{M}}_0$, $\rho_w = 0$ on $\widetilde{M}_0$. Since $\rho_w + \sum_{x\in A}\rho_x = 1$ on M, $\rho_w + \sum_{x\in A}\rho_x = 1$ on $\widetilde{M}_0$ in particular, so because $\rho_w = 0$ on $\widetilde{M}_0$, $\sum_{x\in A}\rho_x = 1$ on $\widetilde{M}_0$.

On all of M, $\sum_{x\in A}\rho_x = 1 - \rho_w \in [0,1]$. □

Proof of Lemma 1. By [24, Proposition 0], since each f_i is analytic and is not identically zero on any connected component of X, the Lebesgue measure of $f_i^{-1}(0)$ restricted to any connected component is 0. Since any open set in $\mathbb{R}^d$ has countably many connected components, the Lebesgue measure of $f_i^{-1}(0)$ is 0. Since I is countable, the Lebesgue measure of $\bigcup_{i\in I} f_i^{-1}(0)$ is 0. □

3.3 Standard form

Before stating the theorem, it is worth pausing to separate the roles of W_0 and X_{res} as in Definition 31. W_0 is the set where at least one f_i vanishes, so only small neighborhoods of W_0 can contribute significantly to the Laplace integrals or affect the meromorphic continuation of the zeta function. When we resolve singularities, we do not only keep track of the analytic factors $\widetilde{f}_i$ of the f_i. We must also keep track of the boundary functions π_s that define W, and, if desired, any chosen finite family of auxiliary analytic functions ϖ_t. For that reason, the bad locus in the charts is the larger set X_{res}, not just W_0: outside X_{res}, the maps g_j are analytic isomorphisms.

The point of the standard form is that, after restricting to an open neighborhood of W_0, all of the geometry can be viewed through finitely many coordinate charts. In each chart g_j, the exceptional

set Γ_j is exactly a union of coordinate hyperplanes, and every relevant analytic quantity becomes a monomial in the coordinates $u_1, \ldots, u_d$ times a unit. For the original functions f_i, this produces a monomial times a positive smooth unit. For the boundary defining functions π_s and the absolute value of the Jacobian determinant, the unit is positive analytic. For the auxiliary analytic functions ϖ_t, the domain of the units on each chart is connected, so the units have constant sign. Meanwhile, the pulled-back weight $\phi \circ g_j$ remains smooth. The functions ψ_j and ρ_j keep track of how these charts are patched together through a partition of unity. This form will help us prove the meromorphic continuation of the zeta function, because once the analytic factors have been reduced to monomials times units, the integral over W can be decomposed through the charts and integrated much more explicitly.

Theorem 7 (Standard form)**.** *(→ proof) Let*

$$\overline{\Delta} = [-1,1]^d, \qquad \Delta^\circ = (-1,1)^d.$$

Assume the main setting of Definition 1 and the notation of Definition 31.

Then there exist:

1. *an open neighborhood $\widetilde{W}_0 \subseteq X$ of W_0;*
2. *a connected open neighborhood $\widetilde{\Delta} \subseteq \mathbb{R}^d$ of $\overline{\Delta}$;*
3. *a number $\epsilon > 0$;*
4. *a finite set $\mathcal{J}$;*

with $\mathcal{J} = \emptyset$ if and only if $W_0 = \emptyset$, such that

$$\{w \in W : \exists i \in [1,k],\ f_i(w) < \epsilon\} \subseteq \widetilde{W}_0.$$

If $W_0 = \emptyset$, then $\widetilde{W}_0 = \emptyset$.

If $W_0 \neq \emptyset$, then $\mathcal{J} \neq \emptyset$. In this case, for each $j \in \mathcal{J}$ there are:

1. *a real analytic map*
$$g_j : \widetilde{\Delta} \to X;$$
2. *the closed set*
$$\Gamma_j := g_j^{-1}(X_{\mathrm{res}}) \subseteq \widetilde{\Delta};$$
3. *the open set*
$$W_j := g_j(\Delta^\circ \setminus \Gamma_j) \subseteq X \setminus X_{\mathrm{res}};$$
4. *smooth functions*
$$\psi_j \in C^\infty(X \setminus X_{\mathrm{res}}, [0,1]), \qquad \rho_j \in C^\infty(\widetilde{\Delta}, [0,1]);$$
5. *positive smooth units*
$$a_{ji} : \widetilde{\Delta} \to \mathbb{R}_{>0};$$

6. *positive analytic units*
$$d_{js}, \widetilde{b}_j : \widetilde{\Delta} \to \mathbb{R}_{>0};$$

7. *for each* $t \in \mathcal{T}$*, an analytic unit*
$$c_{jt} : \widetilde{\Delta} \to \mathbb{R}^{\times};$$

8. *a smooth function*
$$b_j : \widetilde{\Delta} \to \mathbb{R};$$

9. *exponents*
$$k_{jil}, p_{jsl}, r_{jl}, h_{jl} \in \mathbb{Z}_{\geq 0};$$

10. *for each* $t \in \mathcal{T}$*, exponents*
$$n_{jtl} \in \mathbb{Z}_{\geq 0};$$

These objects satisfy:

1. *if*
$$\begin{aligned} M_j := &\{l \in [1,d] : \exists i \in [1,k] \text{ such that } k_{jil} > 0\} \\ &\cup \{l \in [1,d] : \exists s \in [1,m] \text{ such that } p_{jsl} > 0\} \\ &\cup \{l \in [1,d] : \exists t \in \mathcal{T} \text{ such that } n_{jtl} > 0\} \end{aligned}$$
then
$$\begin{aligned} \Gamma_j &= \bigcup_{l \in M_j} (\widetilde{\Delta} \cap \{u_l = 0\}) \\ &= \{u \in \widetilde{\Delta} : \exists i \in [1,k] \text{ such that } (f_i \circ g_j)(u) = 0\} \\ &\quad \cup \{u \in \widetilde{\Delta} : \exists s \in [1,m] \text{ such that } (\pi_s \circ g_j)(u) = 0\} \\ &\quad \cup \{u \in \widetilde{\Delta} : \exists t \in \mathcal{T} \text{ such that } (\varpi_t \circ g_j)(u) = 0\} \end{aligned}$$

2.
$$g_j|_{\widetilde{\Delta} \setminus \Gamma_j} : \widetilde{\Delta} \setminus \Gamma_j \to g_j(\widetilde{\Delta}) \setminus X_{\mathrm{res}}$$
is an analytic isomorphism;

3. $\operatorname{supp} \psi_j \subseteq W_j$*, where the support is taken in the topology of* $X \setminus X_{\mathrm{res}}$*;*

4.
$$\sum_{j \in \mathcal{J}} \psi_j(w) = 1 \quad (w \in (\widetilde{W}_0 \cap W) \setminus W_{\mathrm{res}}), \qquad \sum_{j \in \mathcal{J}} \psi_j(x) \in [0,1] \quad (x \in X \setminus X_{\mathrm{res}});$$

5.
$$\rho_j = \psi_j \circ g_j \qquad \text{on } \widetilde{\Delta} \setminus \Gamma_j;$$

6. $\rho_j = 1$ *on an open neighborhood of* 0*;*

7. *for every* $i \in [1,k]$,

$$f_i \circ g_j = a_{ji} \prod_{l=1}^{d} u_l^{k_{jil}};$$

8. *for every* $s \in [1,m]$,

$$\pi_s \circ g_j = d_{js} \prod_{l=1}^{d} u_l^{p_{jsl}};$$

9. *for every* $t \in \mathcal{T}$,

$$\varpi_t \circ g_j = c_{jt} \prod_{l=1}^{d} u_l^{n_{jtl}};$$

10.

$$|\det Dg_j| = \widetilde{b}_j \prod_{l=1}^{d} |u_l|^{r_{jl}};$$

11.

$$h_{jl} = r_{jl}, \qquad b_j = \widetilde{b}_j(\phi \circ g_j),$$

and

$$|\det Dg_j|(\phi \circ g_j) = b_j \prod_{l=1}^{d} |u_l|^{h_{jl}}.$$

Proof of Theorem 7. Since W is compact and $W_0 \subseteq W$ is closed, W_0 is compact. If $W_0 = \emptyset$, set

$$\widetilde{W}_0 := \emptyset, \qquad \widetilde{\Delta} := (-2,2)^d, \qquad \mathcal{J} := \emptyset.$$

Since $W_0 = \emptyset$, no f_i vanishes on W. Since W is compact and each f_i is continuous, there exists $m_0 > 0$ such that

$$f_i(w) \geq m_0 \qquad (w \in W,\ i \in [1,k]).$$

Set $\epsilon := m_0/2$. Then

$$\{w \in W : \exists i \in [1,k],\ f_i(w) < \epsilon\} = \emptyset \subseteq \widetilde{W}_0.$$

The statement has no j-indexed chart conditions in this case, so the theorem holds.

Assume from now on that $W_0 \neq \emptyset$. For each $x \in W_0$, each π_s is not identically zero around x: otherwise π_s would vanish on an open neighborhood of x, contradicting $x \in W = \overline{W^\circ}$ and $\pi_s > 0$ on W°. $\widetilde{f}_i$ and ϖ_t are not identically zero around x by assumption. Apply Theorem 6 to the family

$$\widetilde{f}_1, \ldots, \widetilde{f}_k, \pi_1, \ldots, \pi_m, \varpi_t \ (t \in \mathcal{T}).$$

This yields a connected open set $W_x \subseteq X$ containing x, a real analytic manifold U_x, and a surjective proper real analytic map

$$g_x : U_x \to W_x$$

such that, with

$$E_x := g_x^{-1}(X_{\mathrm{res}}),$$

the restriction

$$g_x\big|_{U_x \setminus E_x} : U_x \setminus E_x \to W_x \setminus X_{\mathrm{res}}$$

is an analytic isomorphism. Let

$$K_x := g_x^{-1}(x) \cap \overline{g_x^{-1}(W^\circ \cap W_x)}.$$

By Lemma 10, K_x is non-empty and compact.

Fix $x \in W_0$ and $y \in K_x$. By Lemma 11, there exists a connected chart

$$\pi_{x,y} : U_{x,y} \to \widetilde{U}_{x,y} \subseteq \mathbb{R}^d$$

with $\pi_{x,y}(y) = 0$ and

$$\overline{\Delta} \subseteq \widetilde{U}_{x,y},$$

together with positive smooth units

$$a_{x,y,i} : \widetilde{U}_{x,y} \to \mathbb{R}_{>0}$$

and positive analytic units

$$d_{x,y,s}, \widetilde{b}_{x,y} : \widetilde{U}_{x,y} \to \mathbb{R}_{>0}$$

and, for each $t \in \mathcal{T}$, an analytic unit

$$c_{x,y,t} : \widetilde{U}_{x,y} \to \mathbb{R}^\times$$

and exponents

$$k_{x,y,i,l}, p_{x,y,s,l}, r_{x,y,l} \in \mathbb{Z}_{\geq 0}$$

and, for each $t \in \mathcal{T}$, exponents

$$n_{x,y,t,l} \in \mathbb{Z}_{\geq 0}$$

such that, writing

$$\widetilde{g}_{x,y} := g_x \circ \pi_{x,y}^{-1} : \widetilde{U}_{x,y} \to W_x,$$

we have on $\widetilde{U}_{x,y}$,

$$\begin{aligned}
f_i \circ \widetilde{g}_{x,y} &= a_{x,y,i} \prod_{l=1}^{d} u_l^{k_{x,y,i,l}}, \\
\pi_s \circ \widetilde{g}_{x,y} &= d_{x,y,s} \prod_{l=1}^{d} u_l^{p_{x,y,s,l}}, \\
\varpi_t \circ \widetilde{g}_{x,y} &= c_{x,y,t} \prod_{l=1}^{d} u_l^{n_{x,y,t,l}}, \\
|\det D\widetilde{g}_{x,y}| &= \widetilde{b}_{x,y} \prod_{l=1}^{d} |u_l|^{r_{x,y,l}}.
\end{aligned}$$

For $x \in W_0$ and $y \in K_x$, define

$$\chi_{x,y} := \Delta^\circ, \qquad \chi'_{x,y} := \pi_{x,y}^{-1}(\Delta^\circ), \qquad \chi''_{x,y} := g_x(\chi'_{x,y}).$$

Since $0 = \pi_{x,y}(y) \in \Delta^\circ$, each $\chi'_{x,y}$ is an open neighborhood of y. Because K_x is compact, there exists a finite set

$$B_x \subseteq K_x$$

such that

$$K_x \subseteq \bigcup_{y \in B_x} \chi'_{x,y}.$$

Apply Lemma 16 with the manifold U_x, the finite subset B_x, the open sets $\chi'_{x,y}$, and the closed set K_x. We obtain an open set $\widetilde{K}_x \subseteq U_x$ and functions

$$\vartheta_{x,y} \in C^\infty(U_x, [0,1]) \qquad (y \in B_x)$$

such that

$$K_x \subseteq \widetilde{K}_x \subseteq \bigcup_{y \in B_x} \chi'_{x,y},$$

$$\operatorname{supp} \vartheta_{x,y} \subseteq \chi'_{x,y}, \qquad \sum_{y \in B_x} \vartheta_{x,y} = 1 \ \text{ on } \widetilde{K}_x, \qquad \sum_{y \in B_x} \vartheta_{x,y} \in [0,1] \ \text{ on } U_x,$$

$$\vartheta_{x,y} = 1 \quad \text{on an open neighborhood of } y.$$

By Lemma 12, there exists an open set

$$Q_x \subseteq W_x$$

such that

$$x \in Q_x \qquad \text{and} \qquad Q_x \cap W^\circ \subseteq g_x(\widetilde{K}_x).$$

Since $\widetilde{K}_x \subseteq \bigcup_{y \in B_x} \chi'_{x,y}$, we have

$$g_x(\widetilde{K}_x) \subseteq \bigcup_{y \in B_x} \chi''_{x,y}.$$

Since W_0 is compact, there is a finite set $A \subseteq W_0$ such that

$$W_0 \subseteq \bigcup_{x \in A} Q_x.$$

Again apply Lemma 16 to the manifold X, with finite subset A, open sets Q_x, and closed set W_0. We obtain an open set

$$\widetilde{W}_0 \subseteq X$$

and functions

$$\eta_x \in C^\infty(X, [0,1]) \qquad (x \in A)$$

such that

$$W_0 \subseteq \widetilde{W}_0 \subseteq \bigcup_{x \in A} Q_x,$$

$$\operatorname{supp} \eta_x \subseteq Q_x, \qquad \sum_{x \in A} \eta_x = 1 \ \text{ on } \widetilde{W}_0, \qquad \sum_{x \in A} \eta_x \in [0,1] \ \text{ on } X,$$

and

$$\eta_x = 1 \quad \text{on an open neighborhood of } x.$$

Let

$$\Lambda := \{(x,y) : x \in A,\ y \in B_x\}.$$

Since $W_0 \neq \emptyset$, the finite set A is nonempty. Since each K_x is nonempty, each B_x is nonempty. Hence $\Lambda \neq \emptyset$. Now apply Lemma 13. For each $(x,y) \in \Lambda$, this gives functions

$$\psi_{x,y} \in C^\infty(X \setminus X_{\mathrm{res}}, [0,1]), \qquad \widetilde{\rho}_{x,y} \in C^\infty(\widetilde{U}_{x,y}, [0,1]),$$

such that

$$\operatorname{supp} \psi_{x,y} \subseteq \chi''_{x,y} \setminus X_{\mathrm{res}},$$

$$\sum_{x \in A} \sum_{y \in B_x} \psi_{x,y} = 1 \quad \text{on } (\widetilde{W}_0 \cap W) \setminus W_{\mathrm{res}},$$

$$\sum_{x \in A} \sum_{y \in B_x} \psi_{x,y} \in [0,1] \quad \text{on } X \setminus X_{\mathrm{res}},$$

$$\widetilde{\rho}_{x,y} = \psi_{x,y} \circ \widetilde{g}_{x,y} \quad \text{on } \widetilde{U}_{x,y} \setminus \widetilde{g}_{x,y}^{-1}(X_{\mathrm{res}}),$$

and

$$\widetilde{\rho}_{x,y} = 1 \quad \text{on an open neighborhood of } 0.$$

Since Λ is finite and each $\widetilde{U}_{x,y}$ is an open neighborhood of $\overline{\Delta}$, the intersection

$$\bigcap_{(x,y) \in \Lambda} \widetilde{U}_{x,y}$$

is an open neighborhood of $\overline{\Delta}$ in $\mathbb{R}^d$. Restricting further, we obtain a connected open set $\widetilde{\Delta}$ such that

$$\overline{\Delta} \subsetneq \widetilde{\Delta} \subseteq \bigcap_{(x,y) \in \Lambda} \widetilde{U}_{x,y}.$$

For each $(x,y) \in \Lambda$, define

$$g_{x,y} := \widetilde{g}_{x,y}|_{\widetilde{\Delta}} : \widetilde{\Delta} \to X, \qquad \rho_{x,y} := \widetilde{\rho}_{x,y}|_{\widetilde{\Delta}},$$

$$\Gamma_{x,y} := g_{x,y}^{-1}(X_{\mathrm{res}}), \qquad W_{x,y} := g_{x,y}(\Delta^\circ \setminus \Gamma_{x,y}).$$

Restrict the smooth units $a_{x,y,i}$ and the analytic units $d_{x,y,s}, c_{x,y,t}, \widetilde{b}_{x,y}$ from $\widetilde{U}_{x,y}$ to $\widetilde{\Delta}$.

Define

$$\begin{aligned} M_{x,y} := &\{l \in [1,d] : \exists i \in [1,k] \text{ such that } k_{x,y,i,l} > 0\} \\ &\cup \{l \in [1,d] : \exists s \in [1,m] \text{ such that } p_{x,y,s,l} > 0\} \\ &\cup \{l \in [1,d] : \exists t \in \mathcal{T} \text{ such that } n_{x,y,t,l} > 0\} \end{aligned}$$

Since each of $a_{x,y,i}, d_{x,y,s}$ is everywhere positive on $\widetilde{\Delta}$, and each $c_{x,y,t}$ is nowhere-vanishing on $\widetilde{\Delta}$, we have for $u \in \widetilde{\Delta}$,

$$(f_i \circ g_{x,y})(u) = 0 \iff u_l = 0 \text{ for some } l \text{ with } k_{x,y,i,l} > 0,$$

and

$$(\pi_s \circ g_{x,y})(u) = 0 \iff u_l = 0 \text{ for some } l \text{ with } p_{x,y,s,l} > 0.$$

Similarly, for each $t \in \mathcal{T}$,

$$(\varpi_t \circ g_{x,y})(u) = 0 \iff u_l = 0 \text{ for some } l \text{ with } n_{x,y,t,l} > 0.$$

Because

$$X_0 = \bigcup_{i=1}^{k} \{\widetilde{f}_i = 0\}, \qquad X_{\text{res}} = X_0 \cup \bigcup_{s=1}^{m} \{\pi_s = 0\} \cup \bigcup_{t \in \mathcal{T}} \{\varpi_t = 0\},$$

it follows that

$$\begin{aligned}
\Gamma_{x,y} &= \bigcup_{l \in M_{x,y}} (\widetilde{\Delta} \cap \{u_l = 0\}) \\
&= \{u \in \widetilde{\Delta} : \exists i \in [1,k] \text{ such that } (f_i \circ g_{x,y})(u) = 0\} \\
&\quad \cup \{u \in \widetilde{\Delta} : \exists s \in [1,m] \text{ such that } (\pi_s \circ g_{x,y})(u) = 0\} \\
&\quad \cup \{u \in \widetilde{\Delta} : \exists t \in \mathcal{T} \text{ such that } (\varpi_t \circ g_{x,y})(u) = 0\}
\end{aligned}$$

Next, $\pi_{x,y}^{-1}(\widetilde{\Delta}) \setminus E_x$ is an open subset of $U_x \setminus E_x$. Since

$$g_x\big|_{U_x \setminus E_x} : U_x \setminus E_x \to W_x \setminus X_{\text{res}}$$

is an analytic isomorphism, its restriction to

$$\pi_{x,y}^{-1}(\widetilde{\Delta}) \setminus E_x$$

is an analytic isomorphism onto the open set

$$g_x(\pi_{x,y}^{-1}(\widetilde{\Delta})) \setminus X_{\text{res}}$$

Composing with the analytic isomorphism $\pi_{x,y}$, we obtain an analytic isomorphism

$$g_{x,y}\big|_{\widetilde{\Delta} \setminus \Gamma_{x,y}} : \widetilde{\Delta} \setminus \Gamma_{x,y} \to g_{x,y}(\widetilde{\Delta}) \setminus X_{\text{res}}.$$

Similarly, $\pi_{x,y}^{-1}(\Delta^\circ) \setminus E_x$ is open in $U_x \setminus E_x$, so its image

$$g_x(\pi_{x,y}^{-1}(\Delta^\circ) \setminus E_x)$$

is open in $W_x \setminus X_{\text{res}}$, hence in $X \setminus X_{\text{res}}$. Since

$$g_x(\pi_{x,y}^{-1}(\Delta^\circ) \setminus E_x) = g_{x,y}(\Delta^\circ) \setminus X_{\text{res}} = g_{x,y}(\Delta^\circ \setminus \Gamma_{x,y}) = W_{x,y},$$

the set $W_{x,y}$ is open in $X \setminus X_{\text{res}}$. Also,

$$\chi''_{x,y} \setminus X_{\text{res}} = g_x(\chi'_{x,y}) \setminus X_{\text{res}} = g_x(\chi'_{x,y} \setminus E_x) = g_{x,y}(\Delta^\circ \setminus \Gamma_{x,y}) = W_{x,y},$$

and therefore

$$\operatorname{supp} \psi_{x,y} \subseteq W_{x,y}.$$

We have that $\widetilde{\rho}_{x,y} = \psi_{x,y} \circ \widetilde{g}_{x,y}$ on $\widetilde{U}_{x,y} \setminus \widetilde{g}_{x,y}^{-1}(X_{\mathrm{res}})$. This identity restricts to $\widetilde{\Delta}$, so that

$$\rho_{x,y} = \psi_{x,y} \circ g_{x,y} \qquad \text{on } \widetilde{\Delta} \setminus \Gamma_{x,y}.$$

Similarly, since $\widetilde{\rho}_{x,y} = 1$ on an open neighborhood of 0,

$$\rho_{x,y} = 1 \quad \text{on an open neighborhood of } 0.$$

Set

$$C := W \setminus \widetilde{W}_0.$$

Then C is compact because W is assumed compact. Since $W_0 \subseteq \widetilde{W}_0$, C is disjoint from W_0. Since each $f_i \geq 0$ on W in the main setting and $C \cap W_0 = \emptyset$, we have

$$\min_{1 \leq i \leq k} f_i(w) > 0 \qquad (w \in C).$$

Then because C is compact, there exists $\epsilon > 0$ such that

$$\min_{1 \leq i \leq k} f_i(w) \geq \epsilon \qquad (w \in C).$$

Hence, if $w \in W$ and $f_i(w) < \epsilon$ for some i, then $w \notin C$, so

$$w \in \widetilde{W}_0.$$

Therefore,

$$\{w \in W : \exists i \in [1,k],\ f_i(w) < \epsilon\} \subseteq \widetilde{W}_0.$$

Set $\mathcal{J} := \Lambda$. For $j = (x,y) \in \mathcal{J}$, define

$$g_j := g_{x,y}, \qquad \Gamma_j := \Gamma_{x,y}, \qquad W_j := W_{x,y}, \qquad \psi_j := \psi_{x,y}, \qquad \rho_j := \rho_{x,y},$$

and rename the units and exponents by

$$a_{ji}, d_{js}, \widetilde{b}_j, \qquad k_{jil}, p_{jsl}, r_{jl},$$

and, for each $t \in \mathcal{T}$, by

$$c_{jt}, \qquad n_{jtl}.$$

Then all conclusions except the last displayed formula have already been proven. Finally, define

$$h_{jl} := r_{jl}, \qquad b_j := \widetilde{b}_j(\phi \circ g_j).$$

Since $\widetilde{b}_j$ is a positive analytic unit on $\widetilde{\Delta}$, and $\phi \circ g_j$ is smooth on $\widetilde{\Delta}$, b_j is smooth on $\widetilde{\Delta}$. We obtain

$$|\det Dg_j|(\phi \circ g_j) = \widetilde{b}_j(\phi \circ g_j) \prod_{l=1}^{d} |u_l|^{r_{jl}} = b_j \prod_{l=1}^{d} |u_l|^{h_{jl}}.$$

This proves the theorem. □

Definition 32. *Define*

$$\Delta^\times := \Delta^\circ \cap (\mathbb{R}^\times)^d.$$

For each $j \in \mathcal{J}$, set

$$E_j := \left\{ \gamma \in \{\pm 1\}^d : \prod_{l=1}^{d} \gamma_l^{p_{jsl}} = 1 \text{ for every } 1 \le s \le m \right\},$$

and for each $\gamma \in \{\pm 1\}^d$ let

$$O_\gamma^\times := \{u \in \Delta^\times : \operatorname{sign}(u_l) = \gamma_l \text{ for } l = 1, \dots, d\}.$$

We adopt the convention that a sum over the empty set is 0.

Theorem 8. *(→ proof) Assume the setting of Theorem 7 and the definitions in Definition 32. Let m_{Leb} be the Lebesgue measure on $\mathbb{R}^d$. Define*

$$\varphi := \sum_{j \in \mathcal{J}} \psi_j \in C^\infty(X \setminus X_{\mathrm{res}}, [0,1]),$$

and extend φ to W by setting $\varphi = 0$ on W_{res}. Then $\varphi = 1$ on $(\widetilde{W}_0 \cap W) \setminus W_{\mathrm{res}}$. For every $F \in L^1(W, \mathbb{C}, m_{\mathrm{Leb}})$,

$$\int_W F\varphi \, dm_{\mathrm{Leb}} = \sum_{j \in \mathcal{J}} \sum_{\gamma \in E_j} \int_{O_\gamma^\times} (F \circ g_j) \rho_j |\det Dg_j| \, dm_{\mathrm{Leb}}.$$

Consequently,

$$\int_W F \, dm_{\mathrm{Leb}} = \sum_{j \in \mathcal{J}} \sum_{\gamma \in E_j} \int_{O_\gamma^\times} (F \circ g_j) \rho_j |\det Dg_j| \, dm_{\mathrm{Leb}} + \int_{W \setminus \widetilde{W}_0} F \cdot (1 - \varphi) \, dm_{\mathrm{Leb}}.$$

Further, if $F : W \to \mathbb{C}$ is measurable and

$$\sum_{j \in \mathcal{J}} \sum_{\gamma \in E_j} \int_{O_\gamma^\times} (|F| \circ g_j) \rho_j |\det Dg_j| \, dm_{\mathrm{Leb}} + \int_{W \setminus \widetilde{W}_0} |F| \cdot (1 - \varphi) \, dm_{\mathrm{Leb}} < \infty,$$

then $F \in L^1(W, \mathbb{C}, m_{\mathrm{Leb}})$, and the same identities hold for F.

Proof of Theorem 8. Since $\mathcal{J}$ is finite, the function $\varphi = \sum_{j \in \mathcal{J}} \psi_j$ is smooth on $X \setminus X_{\mathrm{res}}$, and its extension to W_{res} is measurable. The results

$$\varphi(x) = \sum_{j \in \mathcal{J}} \psi_j(x) = 1 \quad (x \in (\widetilde{W}_0 \cap W) \setminus W_{\mathrm{res}}) \qquad \varphi(x) = \sum_{j \in \mathcal{J}} \psi_j(x) \in [0,1] \quad (x \in X \setminus X_{\mathrm{res}});$$

are stated in Theorem 7. W_{res} has Lebesgue measure zero by Lemma 1. So it is equivalent to show that for every $F \in L^1(W \setminus W_{\mathrm{res}}, \mathbb{C}, m_{\mathrm{Leb}})$,

1.
$$\int_{W \setminus W_{\mathrm{res}}} F\varphi \, dm_{\mathrm{Leb}} = \sum_{j \in \mathcal{J}} \sum_{\gamma \in E_j} \int_{O_\gamma^\times} (F \circ g_j) \rho_j |\det Dg_j| \, dm_{\mathrm{Leb}}.$$

2.

$$\int_{W\setminus W_{\mathrm{res}}} F\,dm_{\mathrm{Leb}} = \sum_{j\in\mathcal{J}}\sum_{\gamma\in E_j}\int_{O_\gamma^\times}(F\circ g_j)\rho_j|\det Dg_j|\,dm_{\mathrm{Leb}} + \int_{(W\setminus W_{\mathrm{res}})\setminus\widetilde{W}_0} F\cdot(1-\varphi)\,dm_{\mathrm{Leb}}.$$

For any $j\in\mathcal{J}$ and $u\in\Delta^\times$, write

$$\gamma := \mathrm{sign}(u) = (\mathrm{sign}(u_1),\dots,\mathrm{sign}(u_d))\in\{\pm1\}^d.$$

By Theorem 7,

$$\pi_s(g_j(u)) = d_{js}(u)\prod_{l=1}^d u_l^{p_{jsl}},\qquad 1\le s\le m,$$

with $d_{js}(u)>0$. Since every $u_l\neq 0$, each $\pi_s(g_j(u))\neq 0$, and

$$\mathrm{sign}(\pi_s(g_j(u))) = \prod_{l=1}^d \gamma_l^{p_{jsl}}.$$

By Definition 31, $W^\circ=\{x\in X:\pi_s(x)>0 \text{ for every } s\}$. Since $W=\overline{W^\circ}$, every point of W has $\pi_s\ge 0$ for every s. Thus, at points of X where no π_s vanishes, membership in W is equivalent to positivity of every π_s. Therefore

$$g_j(u)\in W \iff \pi_s(g_j(u))>0 \text{ for every } s \iff \prod_{l=1}^d \gamma_l^{p_{jsl}} = 1 \text{ for every } s \iff \gamma\in E_j.$$

It follows that

$$g_j^{-1}(W)\cap\Delta^\times = \bigsqcup_{\gamma\in E_j} O_\gamma^\times.$$

In particular, if $\gamma\in E_j$, then $g_j(O_\gamma^\times)\subseteq W$. Also $O_\gamma^\times\subseteq\Delta^\times\subseteq\Delta^\circ\setminus\Gamma_j$, so

$$g_j(O_\gamma^\times)\subseteq W_j\subseteq X\setminus X_{\mathrm{res}},$$

hence

$$g_j(O_\gamma^\times)\subseteq W\setminus W_{\mathrm{res}}.$$

Now define

$$N_j := ((\Delta^\circ\setminus\Gamma_j)\cap g_j^{-1}(W))\setminus\Delta^\times.$$

Since

$$N_j\subseteq\Delta^\circ\setminus\Delta^\times = \bigcup_{l=1}^d(\Delta^\circ\cap\{u_l=0\}),$$

the set N_j has Lebesgue measure zero by Lemma 1 using the analytic functions u_l on the open set Δ°. Because g_j is real analytic on an open neighborhood of the compact set $\overline{\Delta}$, it is C^1 on $\overline{\Delta}$, hence Lipschitz there. Therefore $g_j(N_j)$ also has Lebesgue measure zero.

By Theorem 7, the map

$$g_j|_{\Delta^\circ\setminus\Gamma_j}$$

is injective. Since

$$W_j = g_j(\Delta^\circ \setminus \Gamma_j),$$

we obtain

$$W \cap W_j = g_j((\Delta^\circ \setminus \Gamma_j) \cap g_j^{-1}(W)) = g_j(N_j) \sqcup \bigsqcup_{\gamma \in E_j} g_j(O_\gamma^\times).$$

Now let $F \in L^1(W \setminus W_{\mathrm{res}}, \mathbb{C}, m_{\mathrm{Leb}})$. Since $\mathcal{J}$ is finite and each ψ_j is bounded,

$$\int_{W \setminus W_{\mathrm{res}}} F\varphi \, dm_{\mathrm{Leb}} = \sum_{j \in \mathcal{J}} \int_{W \setminus W_{\mathrm{res}}} F\psi_j \, dm_{\mathrm{Leb}}.$$

By Theorem 7,

$$\operatorname{supp} \psi_j \subseteq W_j,$$

so

$$\int_{W \setminus W_{\mathrm{res}}} F\psi_j \, dm_{\mathrm{Leb}} = \int_{W \cap W_j} F\psi_j \, dm_{\mathrm{Leb}}.$$

Using the decomposition of $W \cap W_j$ above and the fact that $g_j(N_j)$ has Lebesgue measure zero, we get

$$\int_{W \cap W_j} F\psi_j \, dm_{\mathrm{Leb}} = \sum_{\gamma \in E_j} \int_{g_j(O_\gamma^\times)} F\psi_j \, dm_{\mathrm{Leb}}.$$

For each $\gamma \in E_j$,

$$g_j|_{\widetilde{\Delta} \setminus \Gamma_j} : \widetilde{\Delta} \setminus \Gamma_j \to g_j(\widetilde{\Delta}) \setminus X_{\mathrm{res}}$$

is an analytic isomorphism by Theorem 7. Also, by Theorem 7,

$$\Gamma_j = \bigcup_{l \in M_j} (\widetilde{\Delta} \cap \{u_l = 0\}).$$

Because $O_\gamma^\times$ is disjoint from each $\{u_l = 0\}$, $O_\gamma^\times \subseteq \widetilde{\Delta} \setminus \Gamma_j$. So for each $\gamma \in E_j$, the restriction

$$g_j|_{O_\gamma^\times} : O_\gamma^\times \to g_j(O_\gamma^\times)$$

is an analytic isomorphism. Therefore the change-of-variables formula gives

$$\int_{g_j(O_\gamma^\times)} F\psi_j \, dm_{\mathrm{Leb}} = \int_{O_\gamma^\times} (F \circ g_j)(\psi_j \circ g_j) |\det Dg_j| \, dm_{\mathrm{Leb}}.$$

By Theorem 7,

$$\psi_j \circ g_j = \rho_j \qquad \text{on } \widetilde{\Delta} \setminus \Gamma_j,$$

hence in particular on $O_\gamma^\times$. So

$$\int_{g_j(O_\gamma^\times)} F\psi_j \, dm_{\mathrm{Leb}} = \int_{O_\gamma^\times} (F \circ g_j)\rho_j |\det Dg_j| \, dm_{\mathrm{Leb}}.$$

Summing over j and γ proves

$$\int_{W \setminus W_{\mathrm{res}}} F\varphi \, dm_{\mathrm{Leb}} = \sum_{j \in \mathcal{J}} \sum_{\gamma \in E_j} \int_{O_\gamma^\times} (F \circ g_j)\rho_j |\det Dg_j| \, dm_{\mathrm{Leb}}.$$

For the second identity, write on $W \setminus W_{\mathrm{res}}$,

$$F = F\varphi + F \cdot (1 - \varphi).$$

Since $\varphi = 1$ on $(\widetilde{W}_0 \cap W) \setminus W_{\mathrm{res}}$, we have

$$F \cdot (1 - \varphi) = 0 \qquad \text{on } (\widetilde{W}_0 \cap W) \setminus W_{\mathrm{res}}.$$

Hence

$$\int_{W \setminus W_{\mathrm{res}}} F \cdot (1 - \varphi)\, dm_{\mathrm{Leb}} = \int_{(W \setminus W_{\mathrm{res}}) \setminus \widetilde{W}_0} F \cdot (1 - \varphi)\, dm_{\mathrm{Leb}}.$$

Combining these results,

$$\begin{aligned}
\int_{W \setminus W_{\mathrm{res}}} F\, dm_{\mathrm{Leb}} &= \int_{W \setminus W_{\mathrm{res}}} F\varphi\, dm_{\mathrm{Leb}} + \int_{W \setminus W_{\mathrm{res}}} F \cdot (1 - \varphi)\, dm_{\mathrm{Leb}} \\
&= \int_{W \setminus W_{\mathrm{res}}} F\varphi\, dm_{\mathrm{Leb}} + \int_{(W \setminus W_{\mathrm{res}}) \setminus \widetilde{W}_0} F \cdot (1 - \varphi)\, dm_{\mathrm{Leb}} \\
&= \sum_{j \in \mathcal{J}} \sum_{\gamma \in E_j} \int_{O_\gamma^\times} (F \circ g_j) \rho_j |\det Dg_j|\, dm_{\mathrm{Leb}} + \int_{(W \setminus W_{\mathrm{res}}) \setminus \widetilde{W}_0} F \cdot (1 - \varphi)\, dm_{\mathrm{Leb}}.
\end{aligned}$$

This proves the stated identities for $F \in L^1(W, \mathbb{C}, m_{\mathrm{Leb}})$.

Now let $F : W \to \mathbb{C}$ be measurable. For each $N \in \mathbb{Z}_{\geq 0}$, define

$$F_N := \min(|F|, N).$$

Since W is compact, $F_N \in L^1(W, \mathbb{C}, m_{\mathrm{Leb}})$. So the two identities already proved apply to F_N. So

$$\int_W F_N\, dm_{\mathrm{Leb}} = \sum_{j \in \mathcal{J}} \sum_{\gamma \in E_j} \int_{O_\gamma^\times} (F_N \circ g_j) \rho_j |\det Dg_j|\, dm_{\mathrm{Leb}} + \int_{W \setminus \widetilde{W}_0} F_N \cdot (1 - \varphi)\, dm_{\mathrm{Leb}}.$$

$F_N \to |F|$ pointwise and monotonically as $N \to \infty$. And by assumption,

$$L := \sum_{j \in \mathcal{J}} \sum_{\gamma \in E_j} \int_{O_\gamma^\times} (|F| \circ g_j) \rho_j |\det Dg_j|\, dm_{\mathrm{Leb}} + \int_{W \setminus \widetilde{W}_0} |F| \cdot (1 - \varphi)\, dm_{\mathrm{Leb}} < \infty.$$

Because the index set of pairs (j, γ) is finite, monotone convergence applied termwise gives

$$\lim_{N \to \infty} \sum_{j \in \mathcal{J}} \sum_{\gamma \in E_j} \int_{O_\gamma^\times} (F_N \circ g_j) \rho_j |\det Dg_j|\, dm_{\mathrm{Leb}} + \int_{W \setminus \widetilde{W}_0} F_N \cdot (1 - \varphi)\, dm_{\mathrm{Leb}} = L < \infty.$$

Thus,

$$\lim_{N \to \infty} \int_W F_N\, dm_{\mathrm{Leb}} = L < \infty.$$

Then by the monotone convergence theorem again,

$$\int_W |F|\, dm_{\mathrm{Leb}} = L < \infty.$$

So $F \in L^1(W, \mathbb{C}, m_{\mathrm{Leb}})$, and the identities for F follow from the first part of the theorem. This proves the final claim. □

4 Meromorphic continuation of the zeta function

Assume the main setting of Definition 1. Lemma 17 sets the stage for the main result of this section: showing that ζ meromorphically continues to $\mathbb{C}^k$ and that its polar set is a locally finite hyperplane arrangement with finitely many normal spaces. Fix any auxiliary family of analytic functions $\{\varpi_t\}_{t\in\mathcal{T}}$ as in Definition 31. It only enlarges X_{res} and the exceptional sets in the charts. Taking $\mathcal{T} = \emptyset$ recovers the present zeta function without any extra data.

Lemma 17. *(→ proof) ζ is holomorphic on H^k.*

Proof of Lemma 17. Let $\omega : W \times H^k \to \mathbb{C}$ be defined by

$$\omega(w,z) = \begin{cases} \prod_{i=1}^k f_i(w)^{z_i}\phi(w) & : f_i(w) > 0 \text{ for all } i, \\ 0 & : \text{otherwise.} \end{cases}$$

For fixed $z \in H^k$, the map $w \mapsto \omega(w,z)$ is integrable: W is compact, each f_i and ϕ is bounded on W, and $\mathrm{Re}(z_i) > 0$. The map $z \mapsto \omega(w,z)$ is holomorphic in H^k for all w such that $f_i(w) > 0$ for all $i \in [1,k]$. The zero set of f_i is the same as the zero set of $\widetilde{f_i}$, so Lemma 1 shows $W \cap \bigcup_{i=1}^k f_i^{-1}(0)$ has Lebesgue measure zero, hence $z \mapsto \omega(w,z)$ is holomorphic in H^k for almost all $w \in W$.

There exists an open cover Ω of H^k by pre-compact open sets. For $K \in \Omega$, there are $a, b \in \mathbb{R}$ with $0 < a \le \mathrm{Re}(z_i) \le b$ for all $z \in \overline{K}$ and all i. Since the f_i are non-negative and bounded on W, and ϕ is bounded on W, there is $C_K > 0$ such that

$$\sup_{z\in K} |\omega(w,z)| \le C_K.$$

The constant function C_K is integrable on W because W is compact. We differentiate under the integral sign using Proposition 10 by setting $X \leftarrow W$ and $U \leftarrow H^k$, which shows that ζ is holomorphic in H^k. □

Remark 9. *In the proof of Lemma 17 we used Proposition 10, which we prove later.*

4.1 Analysis background

We need a version of Taylor's theorem in which every term is a monomial in a subset of variables and vanishes to high order in the remaining variables. We discuss in Subsection 4.2 why this is the right version of Taylor's theorem for our purposes. In this subsection, whenever $X \subseteq \mathbb{R}^d$ is a closed axis-aligned rectangle, a function on X is called smooth if it extends smoothly to an open neighborhood of X.

Proposition 9. *(→ proof) Let X be a closed axis-aligned rectangle containing 0. Let $N \in \mathbb{Z}_{\ge 0}$. For $\beta \in \{0,\dots,N\}^d$, define*

$$B(\beta) = \{l \in [1,d] : \beta_l = N\}$$
$$B(\beta)^c = \{l \in [1,d] : \beta_l \ne N\}$$

Let $f : X \to \mathbb{C}$ be smooth. For $l \in [1, d]$ and $n \in \mathbb{Z}_{\geq 0}$, define the normalized n-th Taylor coefficient operator on the l-th variable by

$$\widetilde{T}_{ln} f := \frac{1}{n!} [\partial_{x_l}^n f]\big|_{x_l=0}$$

If $N = 0$, define $\widetilde{R}_{l0} := \mathrm{Id}$. If $N \geq 1$, define the normalized N-th Taylor remainder by

$$\widetilde{R}_{lN} f(x) := \frac{1}{(N-1)!} \int_0^1 [\partial_{x_l}^N f](y + t x_l e_l)(1-t)^{N-1} dt,$$

where $y = (x_1, \ldots, x_{l-1}, 0, x_{l+1}, \ldots, x_d)$ and e_l is the l-th standard basis vector. Define the Taylor operators

$$T_{ln} f := x_l^n \widetilde{T}_{ln} f, \qquad R_{lN} f := x_l^N \widetilde{R}_{lN} f.$$

Define

$$S_{ln}^N := \begin{cases} T_{ln} & : n < N \\ R_{lN} & : n = N \end{cases} \qquad \widetilde{S}_{ln}^N := \begin{cases} \widetilde{T}_{ln} & : n < N \\ \widetilde{R}_{lN} & : n = N \end{cases}$$

There exist $g_\beta : X \to \mathbb{C}$ that are smooth and depend only on the variables with indices $l \in B(\beta)$ such that

$$f(x) = \sum_{\beta \in \{0,\ldots,N\}^d} x^\beta g_\beta(x), \qquad g_\beta = \prod_{l=1}^d \widetilde{S}_{l\beta_l}^N f.$$

Further,

$$x^\beta g_\beta = \prod_{l=1}^d S_{l\beta_l}^N f.$$

Proof of Proposition 9. If $N = 0$, then $\{0, \ldots, N\}^d = \{(0, \ldots, 0)\}$, $B(0, \ldots, 0) = [1, d]$, and $\widetilde{S}_{l0}^0 = \mathrm{Id}$. Thus the statement is the identity $f = \mathrm{Id}\, f$, with

$$g_{(0,\ldots,0)} = f = \prod_{l=1}^d \widetilde{S}_{l0}^0 f.$$

So assume $N \geq 1$.

For each $l \in [1, d]$, the one-variable Taylor expansion in the x_l-variable gives

$$f(x) = \sum_{n=0}^{N-1} x_l^n \widetilde{T}_{ln} f(x) + x_l^N \widetilde{R}_{lN} f(x) = \sum_{n=0}^{N} x_l^n \widetilde{S}_{ln}^N f(x).$$

Because each $\widetilde{S}_{ln}^N$ acts only on the x_l-variable, it commutes with $\widetilde{S}_{l'n'}^N$ and with multiplication by powers of $x_{l'}$ whenever $l \neq l'$. Iterating the preceding decomposition over $l = 1, \ldots, d$ gives

$$f(x) = \sum_{\beta \in \{0,\ldots,N\}^d} x^\beta \Big(\prod_{l=1}^d \widetilde{S}_{l\beta_l}^N f \Big)(x).$$

So it remains to check that

$$g_\beta := \prod_{l=1}^{d} \widetilde{S}^N_{l\beta_l} f$$

is smooth and depends only on the variables with indices $l \in B(\beta)$.

Smoothness is clear for $\widetilde{T}_{ln}$. It also holds for $\widetilde{R}_{lN}$: the integrand in the definition of $\widetilde{R}_{lN} f$ is smooth in (x,t) on $X \times [0,1]$, and differentiating under the t-integral is justified because $[0,1]$ is compact and all mixed partials of f are bounded on compact sets.

Now fix $l \in B(\beta)^c$. Then $\widetilde{S}^N_{l\beta_l} = \widetilde{T}_{l\beta_l}$. Let $P_l \subseteq C^\infty(X)$ be the linear subspace of functions that do not depend on x_l. We have $\widetilde{T}_{l\beta_l}(C^\infty(X)) \subseteq P_l$. If $l' \neq l$, then both $\widetilde{T}_{l'n'}$ and $\widetilde{R}_{l'N}$ preserve P_l because they act only on the $x_{l'}$-variable. Hence $g_\beta \in P_l$. Since this holds for every $l \in B(\beta)^c$, the function g_β depends only on the variables with indices $l \in B(\beta)$.

Finally, for each l,

$$S^N_{l\beta_l} f = x_l^{\beta_l} \widetilde{S}^N_{l\beta_l} f.$$

Using again that the operators for distinct variables commute with the corresponding multiplications, we obtain

$$\prod_{l=1}^{d} S^N_{l\beta_l} f = x^\beta \prod_{l=1}^{d} \widetilde{S}^N_{l\beta_l} f = x^\beta g_\beta.$$

□

We also need a form of differentiation under the integral sign.

Proposition 10. *(→ proof) Let (X, Σ, μ) be a measure space. Let $U \subseteq \mathbb{C}^k$ be open. Let $\omega : X \times U \to \mathbb{C}$ be such that*

1. *$x \to \omega(x,z)$ is integrable for all $z \in U$.*
2. *$z \to \omega(x,z)$ is holomorphic in U for almost all $x \in X$.*
3. *There exists an open cover Ω of U such that for all $K \in \Omega$, there exists $D \in L^1(X)$ such that for almost all $x \in X$, $\sup_{z \in K} |\omega(x,z)| \leq D(x)$.*

Then $F(z) := \int_X \omega(x,z) d\mu(x)$ is holomorphic in U. For all multi-indices $\alpha \in \mathbb{Z}^k_{\geq 0}$, let ∂_z^α denote the operator that takes the α_i-th partial derivative with respect to the i-th coordinate in U. Then

$$\int_X \partial_z^\alpha \omega(x,z) d\mu(x) = \partial_z^\alpha \int_X \omega(x,z) d\mu(x)$$

Proof of Proposition 10. We prove the $k = 1$ case. For general k, apply the one-variable argument on coordinate slices. This gives holomorphy in each variable separately and the corresponding coordinate derivative formulas. Then Hartogs' theorem on separate holomorphicity implies that F is jointly holomorphic. Iterating the one-variable derivative formula then gives the displayed formula for every multi-index. Thus it is sufficient to assume $k = 1$.

Also, because Ω is an open cover of U, it is sufficient to show that for each $K \in \Omega$, F is holomorphic and the derivatives have the desired form within K. Fix $K \in \Omega$. There exists $D \in L^1(X)$ such that for almost all $x \in X$, $\sup_{z \in K} |\omega(x,z)| \leq D(x)$.

Fix $x \in X$ such that $\sup_{z \in K} |\omega(x,z)| \leq D(x)$, which holds for almost all $x \in X$. Let $w \in K$. Because K is open, there exists $r > 0$ such that $B(w,r) \subseteq K$. Let $s \in B(w, r/2)$. Then $B(s, r/2) \subseteq K$. Then by Cauchy's estimate, $|\partial_z^n \omega(x,s)| \leq \frac{n!}{(r/2)^n} D(x)$.

Thus $\partial_z^{n-1}\omega(x,s)$ is Lipschitz within $B(w,r/2)$: for all $s \in B(w,r/2)$, $|\frac{\partial_z^{n-1}\omega(x,s) - \partial_z^{n-1}\omega(x,w)}{s-w}| \leq \frac{n!}{(r/2)^n} D(x)$. We also have that $\lim_{s \to w} \frac{\partial_z^{n-1}\omega(x,s) - \partial_z^{n-1}\omega(x,w)}{s-w} = \partial_z^n \omega(x,w)$.

Then by inductively applying the dominated convergence theorem,

$$\int_X \partial_z^n \omega(x,w) d\mu(x) = \partial_z^n \int_X \omega(x,w) d\mu(x).$$

Since w was arbitrary in K, F is holomorphic and the derivatives have the desired form within K. □

Definition 33. *Let $d, d' \in \mathbb{Z}_{\geq 0}$. For a closed axis-aligned rectangle $X \subseteq \mathbb{R}^d$ containing 0, define a $\mathbb{C}$-algebra $A_{d'}(X)$ as the set of functions $\omega : X \times \mathbb{C}^{d'} \to \mathbb{C}$ such that*

1. *For all $z \in \mathbb{C}^{d'}$, $x \to \omega(x,z)$ is smooth.*
2. *For all $\alpha \in \mathbb{Z}^d_{\geq 0}$ and all $x \in X$, $z \to \partial_x^\alpha \omega(x,z)$ is holomorphic.*
3. *For all $\alpha \in \mathbb{Z}^d_{\geq 0}$ and all compact $E \subseteq \mathbb{C}^{d'}$, $\sup_{x \in X, z \in E} |\partial_x^\alpha \omega(x,z)| < \infty$.*

The purpose of $A_{d'}(X)$ is to state a class of weight functions that is stable under the operations involved in our version of Taylor's theorem in Proposition 9, and can be used to help integrands satisfy the assumptions of our version of differentiation under the integral sign in Proposition 10. Lemmas 18, 20, 19, and 21 show that $A_{d'}(X)$ is preserved by the necessary operations, while Lemma 22 explains the payoff of remaining in $A_{d'}(X)$: when the weight function lies in $A_{d'}(X)$, the corresponding integral is holomorphic on a suitable domain.

Lemma 18. *(→ proof) Let $d, d' \in \mathbb{Z}_{\geq 0}$, and let $X \subseteq \mathbb{R}^d$ be a closed axis-aligned rectangle containing 0.*

1. *For every $\alpha \in \mathbb{Z}^d_{\geq 0}$, the operator ∂_x^α maps $A_{d'}(X)$ to $A_{d'}(X)$.*
2. *Every polynomial in the variables $x_1, \ldots, x_d$ belongs to $A_{d'}(X)$.*
3. *$A_{d'}(X)$ is a $\mathbb{C}$-algebra under pointwise addition and multiplication.*

Proof of Lemma 18. For (1), smoothness in x is preserved by differentiation, and for every multi-index β, $x \mapsto \partial_x^{\alpha+\beta}\omega(x,z)$ is smooth while $z \mapsto \partial_x^{\alpha+\beta}\omega(x,z)$ is holomorphic by definition of $A_{d'}(X)$. The required bounds on compact subsets are also part of the definition.

For (2), if $p(x)$ is a polynomial in the x-variables, then $x \mapsto p(x)$ is smooth, all z-dependence is constant, and every x-derivative is bounded on X.

For (3), let $\omega_1, \omega_2 \in A_{d'}(X)$ and $c \in \mathbb{C}$. The constant function 1 belongs to $A_{d'}(X)$ by (2), so $A_{d'}(X)$ contains a multiplicative identity. Also, for every multi-index α,

$$\partial_x^\alpha(\omega_1 + c\omega_2) = \partial_x^\alpha \omega_1 + c\,\partial_x^\alpha \omega_2,$$

so $\omega_1 + c\omega_2$ is again smooth in x, holomorphic in z, and uniformly bounded on $X \times E$ for every compact $E \subseteq \mathbb{C}^{d'}$.

For multiplication, fix a multi-index α. By the higher-order product rule,

$$\partial_x^\alpha(\omega_1\omega_2) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial_x^\beta \omega_1 \partial_x^{\alpha-\beta} \omega_2.$$

Each summand is smooth in x, holomorphic in z, and uniformly bounded on $X \times E$ for every compact $E \subseteq \mathbb{C}^{d'}$, so $\omega_1\omega_2 \in A_{d'}(X)$. □

Lemma 19. *(→ proof) Let $d, d' \in \mathbb{Z}_{\ge 0}$, let $X \subseteq \mathbb{R}^d$ be a closed axis-aligned rectangle containing 0, let $\alpha : X \to \mathbb{R}_{>0}$ be smooth, and fix $m \in [1, d']$. Then*

$$\omega(x, q) := \alpha(x)^{q_m}$$

belongs to $A_{d'}(X)$.

Proof of Lemma 19. Because $\alpha > 0$, the function $\log \alpha$ is smooth on an open neighborhood of X. Write

$$\omega(x, q) = e^{q_m \log \alpha(x)}.$$

For each multi-index β, repeated differentiation in the x-variables shows that $\partial_x^\beta \omega(x, q)$ is a finite sum of terms of the form

$$P_\beta(q_m)\, h_\beta(x)\, e^{q_m \log \alpha(x)},$$

where P_β is a polynomial and h_β is smooth on an open neighborhood of X. Hence $x \mapsto \omega(x, q)$ is smooth for every q, and $q \mapsto \partial_x^\beta \omega(x, q)$ is holomorphic for every x.

Let $E \subseteq \mathbb{C}^{d'}$ be compact. Then q_m ranges over a compact subset of $\mathbb{C}$ as q ranges over E. Since X is compact and $\log \alpha$ and all h_β are smooth on an open neighborhood of X, the factors $h_\beta(x)$ and $e^{q_m \log \alpha(x)}$ are uniformly bounded on $X \times E$. Therefore $\sup_{x \in X,\ q \in E} |\partial_x^\beta \omega(x, q)| < \infty$ for every β. So $\omega \in A_{d'}(X)$. □

Lemma 20. *(→ proof) Let $d, d' \in \mathbb{Z}_{\ge 0}$, and let $X \subseteq \mathbb{R}^d$ be a closed axis-aligned rectangle containing 0. For all $N \in \mathbb{Z}_{\ge 0}$, $l \in [1, d]$, and $n \in [0, N]$, the operators $\widetilde{T}_{ln}$, $\widetilde{R}_{lN}$, $\widetilde{S}^N_{ln}$, T_{ln}, R_{lN}, and S^N_{ln} map $A_{d'}(X)$ to $A_{d'}(X)$.*

Proof of Lemma 20. By Lemma 18, $\partial_{x_l}^n$ maps $A_{d'}(X)$ to $A_{d'}(X)$. Evaluating at $x_l = 0$ preserves smoothness in the remaining variables, preserves holomorphic dependence on z, and preserves the uniform bounds required in Definition 33. Hence $\widetilde{T}_{ln}$ maps $A_{d'}(X)$ to $A_{d'}(X)$.

If $N = 0$, then $\widetilde{R}_{l0} = \mathrm{Id}$, so it also preserves $A_{d'}(X)$. Assume $N \geq 1$, let $\omega \in A_{d'}(X)$, and write $y = (x_1, \ldots, x_{l-1}, 0, x_{l+1}, \ldots, x_d)$. Then

$$(\widetilde{R}_{lN}\omega)(x, z) = \frac{1}{(N-1)!} \int_0^1 [\partial_{x_l}^N \omega](y + t x_l e_l, z)(1-t)^{N-1} dt,$$

where e_l is the l-th standard basis vector. For each multi-index α, differentiating under the t-integral gives a finite linear combination of terms of the form

$$t^m \, (\partial_x^\gamma \omega)(y + t x_l e_l, z)$$

with $m \in \mathbb{Z}_{\geq 0}$ and $\gamma \in \mathbb{Z}_{\geq 0}^d$. Because $(x, t) \mapsto y + t x_l e_l$ maps $X \times [0,1]$ into X, each such term is smooth in x, holomorphic in z, and uniformly bounded on $X \times E \times [0,1]$ for every compact $E \subseteq \mathbb{C}^{d'}$. Therefore $\widetilde{R}_{lN}\omega \in A_{d'}(X)$. The claim for $\widetilde{S}_{ln}^N$ follows from its definition.

By Lemma 18, multiplication by x_l^n preserves $A_{d'}(X)$. Since $T_{ln} = x_l^n \widetilde{T}_{ln}$ and $R_{lN} = x_l^N \widetilde{R}_{lN}$, the same holds for T_{ln} and R_{lN}. The claim for S_{ln}^N again follows from its definition. □

Lemma 21. *(→ proof) Let $d, d' \in \mathbb{Z}_{\geq 0}$, let $X = \prod_{l=1}^d [0, a_l] \subseteq \mathbb{R}^d$ be a closed axis-aligned rectangle containing 0, let $S \subseteq [1, d]$, and let $S^c = [1, d] \setminus S$. Suppose $\omega \in A_{d'}(X)$ does not depend on the variables indexed by S. Then there exists a unique function $\vartheta : X_{S^c} \times \mathbb{C}^{d'} \to \mathbb{C}$, where $X_{S^c} := \prod_{l \in S^c} [0, a_l]$, such that*

$$\omega(x, z) = \vartheta(x_{S^c}, z).$$

Moreover, $\vartheta \in A_{d'}(X_{S^c})$.

Proof of Lemma 21. Uniqueness and existence are immediate because ω is constant along the coordinates indexed by S. For a multi-index $\alpha \in \mathbb{Z}_{\geq 0}^{S^c}$, the derivative $\partial_{x_{S^c}}^\alpha \vartheta$ is obtained by taking the corresponding x-derivative of ω. Therefore smoothness in the remaining variables, holomorphic dependence on z, and the uniform bounds on compact subsets all follow directly from the corresponding properties of ω. □

Lemma 22. *(→ proof) Let $r, d' \in \mathbb{Z}_{\geq 0}$ with $d' \geq r$, let $\omega \in A_{d'}([0,1]^r)$, and let $N \in \mathbb{Z}_{\geq 0}$. Define*

$$U_{r,N} := \{q \in \mathbb{C}^{d'} : Re(q_l + N + 1) > 0 \ \forall l \in [1, r]\}.$$

Then

$$F(q) := \int_{[0,1]^r} \prod_{l=1}^r u_l^{q_l + N} \omega(u, q) du$$

is holomorphic on $U_{r,N}$.

Proof of Lemma 22. We verify the hypotheses of Proposition 10 on $[0,1]^r \times U_{r,N}$.

For each $q \in U_{r,N}$, the function $u \mapsto \prod_{l=1}^r u_l^{q_l+N} \omega(u, q)$ is integrable on $[0,1]^r$ because $\omega(\cdot, q)$ is bounded on $[0,1]^r$ by Definition 33, and $\prod_{l=1}^r u_l^{q_l+N}$ is integrable when $\mathrm{Re}(q_l + N + 1) > 0$ for all $l \in [1, r]$.

For every $u \in (0,1]^r$, the map $q \mapsto \prod_{l=1}^r u_l^{q_l+N}\omega(u,q)$ is holomorphic on $U_{r,N}$ because both factors are holomorphic there.

Let $K \subseteq U_{r,N}$ be compact. Since each coordinate projection of K has real part bounded below by a number strictly larger than $-N-1$, there exists $\epsilon > 0$ such that $\mathrm{Re}(q_l + N) \geq -1 + \epsilon$ for all $q \in K$ and all $l \in [1,r]$. By Definition 33, there exists $C_K > 0$ such that $|\omega(u,q)| \leq C_K$ for all $(u,q) \in [0,1]^r \times K$. Hence

$$\Big|\prod_{l=1}^r u_l^{q_l+N}\omega(u,q)\Big| \leq C_K \prod_{l=1}^r u_l^{-1+\epsilon}$$

for all $(u,q) \in [0,1]^r \times K$, and the right-hand side is integrable on $[0,1]^r$.

Proposition 10 now implies that F is holomorphic on $U_{r,N}$. □

4.2 Toy zeta function

After applying Theorem 8 to view each analytic function as a monomial up to unit, each zeta function is a finite sum of integrals of the form

$$\int_{O_\gamma^\times} \prod_{l=1}^d |u_l|^{\sum_{i=1}^k z_i k_{jil} + h_{jl}} \Big(\prod_{i=1}^k a_{ji}(u)^{z_i}\Big)\rho_j(u)b_j(u)du$$

where $O_\gamma^\times$ is an orthant as in Definition 32, $j \in \mathcal{J}$, and the functions a_{ji}, ρ_j, b_j and the integers $k_{jil}, h_{jl} \in \mathbb{Z}_{\geq 0}$ are as in Theorem 7. In particular, the functions a_{ji} are positive smooth units on an open neighborhood of $[0,1]^d$, while ρ_j and b_j are smooth on an open neighborhood of $[0,1]^d$. By Lemmas 18 and 19, the non-monomial factor can be packaged into a function in $A_{d'}([0,1]^d)$. Up to a holomorphic change of variables, a set of measure zero, and packaging this non-monomial factor into a single function, this is of the form

$$\int_{[0,1]^d} \prod_{l=1}^d u_l^{q_l}\omega(u,q)du$$

for some $\omega \in A_{d'}([0,1]^d)$, where we set $d' = d + k$. Explicitly, we set

$$q = (q_1, \ldots, q_d, q_{d+1}, \ldots, q_{d'}) \in \mathbb{C}^{d'},$$

where

$$q_l = \sum_{i=1}^k z_i k_{jil} + h_{jl} \qquad l \in [1,d],$$

and

$$(q_{d+1}, \ldots, q_{d'}) = z.$$

The point of adjoining these last k coordinates is that the non-monomial factor in our integrand depends on z, and $z \to (q_1, \ldots, q_d)$ might not have a left-inverse. To study the $k = 1$ case, the proof in [3, Lemma 7.3] considers a toy problem similar to this, except that our non-monomial part depends on complex variables as a result of allowing $k > 1$. As explained in Remark 7, when $k = 1$ this dependence can be eliminated. They use a similar Taylor expansion to show that the toy

zeta function meromorphically continues to $\mathbb{C}^{d'}$. Our proof differs in that we show more detail by defining the algebra in Definition 33 to justify differentiation under the integral sign after performing manipulations on the weight functions. We also obtain an explicit formula locally around poles.

The strategy is to expand ω in a Taylor series in the variables $u_1, \ldots, u_d$ to finite order. Each $u_l^{\beta_l}$ from a term of the Taylor series reduces the entire dependence of that term on the u_l variable to a monomial. This contributes an explicit factor $\int_0^1 u_l^{q_l+\beta_l}\, du_l = (q_l+\beta_l+1)^{-1}$, which reveals the poles.

The challenge is that the remainder terms in the Taylor expansion could potentially not meromorphically continue. If we used a more common form of Taylor's theorem such that the remainder terms vanish to total order N across each variable, then we would have terms like $u_1^{N-1}u_2^1$, which would only allow us to conclude that the remainder integral

$$\int_{[0,1]^2} u_1^{q_1+N-1} u_2^{q_2+1} \omega(u,q) du$$

is holomorphic on $\{q \in \mathbb{C}^{d'} : \mathrm{Re}(q_1+N) > 0, \mathrm{Re}(q_2+2) > 0\}$. This does not eventually contain any compact subset of $\mathbb{C}^{d'}$, which would obstruct the proof.

The solution is to use the form of Taylor's theorem in Proposition 9 in which every variable in every term either contributes $u_l^{\beta_l}$ or vanishes to high order. For the variables that vanish to order N, we have factors of the form $u_l^{q_l+N}$, which are still integrable against a smooth weight when $\mathrm{Re}(q_l+N+1) > 0$. Then by differentiating under the integral sign, with $d' \geq d$, we can show that the remainder terms are meromorphic on $\{q \in \mathbb{C}^{d'} : \forall l \in [1,d]\ \mathrm{Re}(q_l+N+1) > 0\}$, which eventually contains any compact subset of $\mathbb{C}^{d'}$.

Definition 34. *For $C \subseteq [1,d]$, write $C = \{c_1 < \cdots < c_m\}$. We write $[0,1]^C$ for the product of the coordinates indexed by C, so $u_C = (u_{c_1}, \ldots, u_{c_m})$ and $du_C = du_{c_1} \cdots du_{c_m}$. If $C = \emptyset$, then $[0,1]^C$ is a point and du_C is the unit measure.*

Proposition 11. *(→ proof) Let $d' \geq d$. Let $\omega \in A_{d'}([0,1]^d)$. Define $\widetilde{\zeta} : H^{d'} \to \mathbb{C}$ by*

$$\widetilde{\zeta}(q) := \int_{[0,1]^d} \prod_{l=1}^{d} u_l^{q_l} \omega(u,q) du.$$

$\widetilde{\zeta}$ meromorphically continues to $\mathbb{C}^{d'}$. The polar set is a union of hyperplanes of the form

$$\{q \in \mathbb{C}^{d'} : q_l + \beta + 1 = 0\} \qquad \beta \in \mathbb{Z}_{\geq 0}.$$

Further, for all $N \in \mathbb{Z}_{\geq 0}$, there exists for each $\beta \in \{0, \ldots, N\}^d$ a function $\vartheta_\beta \in A_{d'}([0,1]^{B(\beta)})$ such that

$$\widetilde{\zeta}(q) = \sum_{\beta \in \{0,\ldots,N\}^d} \frac{\int_{[0,1]^{B(\beta)}} \prod_{l \in B(\beta)} u_l^{q_l+N} \vartheta_\beta(u_{B(\beta)}, q) du_{B(\beta)}}{\prod_{l \in B(\beta)^c} (q_l + \beta_l + 1)},$$

over $\{q \in \mathbb{C}^{d'} : Re(q_l+N+1) > 0\ \forall l \in [1,d]\}$. Further,

$$\int_{[0,1]^{B(\beta)}} \Big| \prod_{l \in B(\beta)} u_l^{q_l+N} \vartheta_\beta(u_{B(\beta)}, q) \Big| du_{B(\beta)} < \infty,$$

over $\{q \in \mathbb{C}^{d'} : Re(q_l + N + 1) > 0 \ \forall l \in B(\beta)\}$. *Explicitly, if*

$$\theta_\beta := \prod_{l=1}^{d} \widetilde{S}^N_{l\beta_l}\omega,$$

then

$$\theta_\beta(u, q) = \vartheta_\beta(u_{B(\beta)}, q).$$

Let $S \subseteq [1, d]$. *Let* $S^c = [1, d] \setminus S$. *Let* $\beta \in \mathbb{Z}^S_{\geq 0}$. *Let*

$$Z := \{q \in \mathbb{C}^{d'} : q_l + \beta_l + 1 = 0 \ \forall l \in S, Re(q_l + 1) > 0 \ \forall l \in S^c\}.$$

There exists $\vartheta_\beta : [0, 1]^{S^c} \times \mathbb{C}^{d'} \to \mathbb{C}$ *and* ξ *meromorphic on* $\mathbb{C}^{d'}$ *such that in an open neighborhood of* Z,

$$\widetilde{\zeta}(q) = \frac{1}{\prod_{l\in S}(q_l + \beta_l + 1)}\Big[\int_{[0,1]^{S^c}} \Big(\prod_{l\in S^c} u_l^{q_l}\Big)\vartheta_\beta(u_{S^c}, q)du_{S^c}\Big] + \xi(q),$$

and $\big(\prod_{l\in S}(q_l + \beta_l + 1)\big)\xi$ *vanishes on* Z. *Explicitly,*

$$\vartheta_\beta = \Big[\prod_{l\in S} \frac{\partial^{\beta_l}_{u_l}}{\beta_l!}\omega\Big]\Big|_{u_l=0 \ \forall l\in S}.$$

Proof of Proposition 11. Fix $N \in \mathbb{Z}_{\geq 0}$. For $\beta \in \{0, \ldots, N\}^d$, define

$$\theta_\beta := \prod_{l=1}^{d} \widetilde{S}^N_{l\beta_l}\omega.$$

By Proposition 9,

$$\omega(u, q) = \sum_{\beta\in\{0,\ldots,N\}^d} u^\beta \theta_\beta(u, q).$$

Each θ_β lies in $A_{d'}([0, 1]^d)$ because each $\widetilde{S}^N_{l\beta_l}$ preserves $A_{d'}([0, 1]^d)$ by Lemma 20. Again by Proposition 9, θ_β does not depend on the variables indexed by $B(\beta)^c$. Therefore there exists $\vartheta_\beta : [0, 1]^{B(\beta)} \times \mathbb{C}^{d'} \to \mathbb{C}$ such that

$$\theta_\beta(u, q) = \vartheta_\beta(u_{B(\beta)}, q).$$

By Lemma 21, $\vartheta_\beta \in A_{d'}([0, 1]^{B(\beta)})$.

Next we explicitly integrate over $[0,1]^{B(\beta)^c}$.

$$\begin{aligned}
&\int_{[0,1]^d} \prod_{l=1}^{d} u_l^{q_l} \omega(u,q) du \\
&= \sum_{\beta \in \{0,\dots,N\}^d} \int_{[0,1]^d} \prod_{l=1}^{d} u_l^{q_l} u^{\beta} \theta_\beta(u,q) du \\
&= \sum_{\beta \in \{0,\dots,N\}^d} \Big[\int_{[0,1]^{B(\beta)^c}} \prod_{l \in B(\beta)^c} u_l^{q_l+\beta_l} du_{B(\beta)^c} \Big] \Big[\int_{[0,1]^{B(\beta)}} \prod_{l \in B(\beta)} u_l^{q_l+\beta_l} \vartheta_\beta(u_{B(\beta)}, q) du_{B(\beta)} \Big] \\
&= \sum_{\beta \in \{0,\dots,N\}^d} \frac{1}{\prod_{l \in B(\beta)^c} (q_l + \beta_l + 1)} \Big[\int_{[0,1]^{B(\beta)}} \prod_{l \in B(\beta)} u_l^{q_l+N} \vartheta_\beta(u_{B(\beta)}, q) du_{B(\beta)} \Big]
\end{aligned}$$

Define $H_{B(\beta),N} := \{q \in \mathbb{C}^{d'} : \mathrm{Re}(q_l + N + 1) > 0 \ \forall l \in B(\beta)\}$. Since $\vartheta_\beta \in A_{d'}([0,1]^{B(\beta)})$, Lemma 22 implies that

$$\int_{[0,1]^{B(\beta)}} \prod_{l \in B(\beta)} u_l^{q_l+N} \vartheta_\beta(u_{B(\beta)}, q) du_{B(\beta)}$$

is holomorphic in $H_{B(\beta),N}$. Since this decomposition holds for every $N \in \mathbb{Z}_{\geq 0}$, for every compact $E \subseteq \mathbb{C}^{d'}$ we may choose N large enough so that $E \subseteq H_{B(\beta),N}$ for every $\beta \in \{0,\dots,N\}^d$. This completes the proof that $\widetilde{\zeta}$ extends meromorphically to $\mathbb{C}^{d'}$ and its polar set has the desired form.

Next we compute coefficients. Since each $\widetilde{T}_{l\beta_l}$ preserves $A_{d'}([0,1]^d)$ by Lemma 20, the function

$$\theta_\beta := \Big(\prod_{l \in S} \widetilde{T}_{l\beta_l} \Big) \omega$$

belongs to $A_{d'}([0,1]^d)$ and does not depend on the variables indexed by S. Explicitly,

$$\theta_\beta = \Big[\prod_{l \in S} \frac{\partial_{u_l}^{\beta_l}}{\beta_l!} \omega \Big] \Big|_{u_l = 0 \ \forall l \in S}.$$

Let $\vartheta_\beta : [0,1]^{S^c} \times \mathbb{C}^{d'}$ be such that $\vartheta_\beta(u_{S^c}, q) = \theta_\beta(u,q)$. By Lemma 21, $\vartheta_\beta \in A_{d'}([0,1]^{S^c})$. Thus

$$\vartheta_\beta = [\prod_{l \in S} \frac{\partial_{u_l}^{\beta_l}}{\beta_l!} \omega] \big|_{u_l = 0 \ \forall l \in S}.$$

Also,

$$\Big(\prod_{l \in S} T_{l\beta_l} \Big) \omega = \Big(\prod_{l \in S} u_l^{\beta_l} \Big) \theta_\beta.$$

Now we compute

$$\begin{aligned}
&\int_{[0,1]^d} \prod_{l=1}^{d} u_l^{q_l} [\prod_{l\in S} T_{l\beta_l}\omega](u,q)du \\
&= [\int_{[0,1]^S} \prod_{l\in S} u_l^{q_l+\beta_l} du_S][\int_{[0,1]^{S^c}} (\prod_{l\in S^c} u_l^{q_l})\vartheta_\beta(u_{S^c},q)du_{S^c}] \\
&= \frac{1}{\prod_{l\in S}(q_l+\beta_l+1)}[\int_{[0,1]^{S^c}} (\prod_{l\in S^c} u_l^{q_l})\vartheta_\beta(u_{S^c},q)du_{S^c}],
\end{aligned}$$

and Lemma 22 shows that the bracketed integral is holomorphic on

$$\{q\in\mathbb{C}^{d'} : \mathrm{Re}(q_l+1)>0\ \forall l\in S^c\}.$$

If $S=\emptyset$, the formula holds with $\xi=0$ on

$$\{q\in\mathbb{C}^{d'} : \mathrm{Re}(q_l+1)>0\ \forall l\}.$$

Assume $S\neq\emptyset$. Choose $N>\max_{l\in S}\beta_l$. For each $l\in S$, write

$$\mathrm{Id}=\sum_{n=0}^{N-1} T_{ln}+R_{lN}=T_{l\beta_l}+\sum_{\substack{0\le n<N\\ n\neq\beta_l}} T_{ln}+R_{lN}.$$

For $l\in S$, put

$$\mathcal{A}_l:=\{T_{ln}: 0\le n<N,\ n\neq\beta_l\}\cup\{R_{lN}\}.$$

Then, multiplying the preceding decompositions over $l\in S$,

$$\begin{aligned}
\mathrm{Id} &= \prod_{l\in S}\left(T_{l\beta_l}+\sum_{A\in\mathcal{A}_l} A\right) \\
&= \prod_{l\in S} T_{l\beta_l}+\sum_{\emptyset\neq C\subseteq S}\ \sum_{\substack{\text{choose } A_l\in\mathcal{A}_l\\ \text{for each } l\in C}} \left(\prod_{l\in C} A_l\right)\left(\prod_{l\in S\setminus C} T_{l\beta_l}\right).
\end{aligned}$$

In the double sum, C is the nonempty set of indices where we choose an operator different from $T_{l\beta_l}$, and for each $l\in C$, A_l is chosen from $\mathcal{A}_l$. The first term, $\prod_{l\in S} T_{l\beta_l}$, is the term where we choose $T_{l\beta_l}$ for every $l\in S$; it gives the displayed fraction with denominator $\prod_{l\in S}(q_l+\beta_l+1)$. Every summand in the double sum has at least one index $l_0\in S$ where either $T_{l_0 n}$ with $n\neq\beta_{l_0}$ or $R_{l_0 N}$ occurs. In the first case its possible denominator $q_{l_0}+n+1$ is nonzero on Z; in the second case the corresponding integral is holomorphic near Z because $N>\beta_{l_0}$. Thus, after multiplying by $\prod_{l\in S}(q_l+\beta_l+1)$, each such term vanishes on Z. Hence there exists ξ meromorphic on $\mathbb{C}^{d'}$ such that in an open neighborhood of Z,

$$\widetilde{\zeta}(q)=\frac{1}{\prod_{l\in S}(q_l+\beta_l+1)}[\int_{[0,1]^{S^c}} (\prod_{l\in S^c} u_l^{q_l})\vartheta_\beta(u_{S^c},q)du_{S^c}]+\xi(q),$$

and $(\prod_{l\in S}(q_l+\beta_l+1))\xi$ vanishes on Z. □

4.3 Meromorphic continuation of the zeta function

Now we prove the main result of this section, the meromorphic continuation of the zeta function from Definition 4. Here is a sketch. Away from the set $\widetilde{W}_0$ from Theorem 7, every f_i is bounded away from 0. So the remainder term from Theorem 8,

$$\int_{W\setminus\widetilde{W}_0}\prod_{i=1}^{k} f_i(w)^{z_i}\phi(w)(1-\varphi(w))dw,$$

is entire in z by Proposition 10. Therefore the only potential poles of ζ come from the finitely many charts in Theorem 7.

Those charts contribute the finite sum of other terms from Theorem 8. In Subsection 4.2, we discussed how to reduce these terms to the toy zeta function, which yields the meromorphic continuation of each term.

Theorem 9 (Meromorphic continuation of the zeta function)**.** *(→ proof) Assume the main setting of Definition 1 and let ζ be as in Definition 4. Fix the auxiliary family $\{\varpi_t\}_{t\in\mathcal{T}}$ from Definition 31, and let the notation from Definition 31, Theorem 7, Definition 32, and Theorem 8 be determined by this choice. Then ζ meromorphically continues to $\mathbb{C}^k$. The polar set is a possibly empty union, over $j\in\mathcal{J}$, $l\in[1,d]$, and $\beta\in\mathbb{Z}_{\geq 0}$, of affine hyperplanes of the form*

$$\{z\in\mathbb{C}^k : \sum_{i=1}^{k} z_i k_{jil} + h_{jl} + \beta + 1 = 0\}.$$

Define for each $j\in\mathcal{J}$ and $\gamma\in E_j$,

$$\zeta_{j\gamma}(z) := \int_{O_\gamma^\times}\prod_{l=1}^{d}|u_l|^{\sum_{i=1}^{k} z_i k_{jil}+h_{jl}}\Big(\prod_{i=1}^{k} a_{ji}(u)^{z_i}\Big)\rho_j(u)b_j(u)du,$$

and define

$$\eta(z) := \int_{W\setminus\widetilde{W}_0}\prod_{i=1}^{k} f_i(w)^{z_i}\phi(w)(1-\varphi(w))dw.$$

Then each $\zeta_{j\gamma}$ meromorphically continues to $\mathbb{C}^k$, η is entire, and

$$\zeta = \sum_{j\in\mathcal{J}}\sum_{\gamma\in E_j}\zeta_{j\gamma} + \eta.$$

For each $j\in\mathcal{J}$ and $\gamma\in E_j$, set $d' := d+k$, and let $A_j:\mathbb{C}^k\to\mathbb{C}^{d'}$ be the affine map defined by

$$(A_j(z))_l = \sum_{i=1}^{k} z_i k_{jil} + h_{jl} \quad (l\in[1,d]), \qquad (A_j(z))_{d+i} = z_i \quad (i\in[1,k]).$$

Using the indexed-cube notation of Definition 34, for all $N\in\mathbb{Z}_{\geq 0}$, there exists for each $\beta\in\{0,\dots,N\}^d$ a function $\widetilde{\vartheta}_{j\gamma\beta}\in A_{d'}([0,1]^{B(\beta)})$ such that, with $\vartheta_{j\gamma\beta}(u,z) := \widetilde{\vartheta}_{j\gamma\beta}(u,A_j(z))$,

$$\zeta_{j\gamma}(z) = \sum_{\beta\in\{0,\dots,N\}^d}\frac{\int_{[0,1]^{B(\beta)}}\prod_{l\in B(\beta)} u_l^{\sum_{i=1}^{k} z_i k_{jil}+h_{jl}+N}\vartheta_{j\gamma\beta}(u_{B(\beta)},z)du_{B(\beta)}}{\prod_{l\in B(\beta)^c}(\sum_{i=1}^{k} z_i k_{jil}+h_{jl}+\beta_l+1)},$$

over $\{z \in \mathbb{C}^k : Re(\sum_{i=1}^k z_i k_{jil} + h_{jl} + N + 1) > 0 \ \forall l \in [1,d]\}$. *Further,*

$$\int_{[0,1]^{B(\beta)}} | \prod_{l \in B(\beta)} u_l^{\sum_{i=1}^k z_i k_{jil} + h_{jl} + N} \vartheta_{j\gamma\beta}(u_{B(\beta)}, z) | du_{B(\beta)} < \infty,$$

over $\{z \in \mathbb{C}^k : Re(\sum_{i=1}^k z_i k_{jil} + h_{jl} + N + 1) > 0 \ \forall l \in B(\beta)\}$. *Explicitly,*

$$\widetilde{\omega}_{j\gamma}(u,q) := \Big(\prod_{i=1}^k a_{ji}(\gamma u)^{q_{i+d}}\Big)\rho_j(\gamma u) b_j(\gamma u)$$

belongs to $A_{d'}([0,1]^d)$. *If*

$$\widetilde{\theta}_{j\gamma\beta} := \prod_{l=1}^d \widetilde{S}^N_{l\beta_l} \widetilde{\omega}_{j\gamma},$$

then

$$\widetilde{\theta}_{j\gamma\beta}(u,q) = \widetilde{\vartheta}_{j\gamma\beta}(u_{B(\beta)}, q),$$
$$\vartheta_{j\gamma\beta}(u,z) = \widetilde{\vartheta}_{j\gamma\beta}(u, A_j(z)).$$

Let $S \subseteq [1,d]$. *Let* $S^c = [1,d] \setminus S$. *Let* $\beta \in \mathbb{Z}^S_{\geq 0}$. *Let*

$$Z = \{z \in \mathbb{C}^k : \sum_{i=1}^k z_i k_{jil} + h_{jl} + \beta_l + 1 = 0 \ \forall l \in S, Re(\sum_{i=1}^k z_i k_{jil} + h_{jl} + 1) > 0 \ \forall l \in S^c\}.$$

There exists $\vartheta_{j\gamma\beta} : [0,1]^{S^c} \times \mathbb{C}^k \to \mathbb{C}$ *and* ξ *meromorphic on* $\mathbb{C}^k$ *such that on an open neighborhood of* Z,

$$\zeta_{j\gamma}(z) = \frac{1}{\prod_{l \in S}(\sum_{i=1}^k z_i k_{jil} + h_{jl} + \beta_l + 1)} \Big[\int_{[0,1]^{S^c}} \Big(\prod_{l \in S^c} u_l^{\sum_{i=1}^k z_i k_{jil} + h_{jl}} \Big) \vartheta_{j\gamma\beta}(u_{S^c}, z) du_{S^c} \Big] + \xi(z),$$

and $(\prod_{l \in S}(\sum_{i=1}^k z_i k_{jil} + h_{jl} + \beta_l + 1))\xi$ *vanishes on* Z. *Explicitly,*

$$\vartheta_{j\gamma\beta} = \Big[\prod_{l \in S} \frac{\partial^{\beta_l}_{u_l}}{\beta_l!} \omega_{j\gamma}\Big]\Big|_{u_l = 0 \ \forall l \in S}$$
$$\omega_{j\gamma}(u,z) = \Big(\prod_{i=1}^k a_{ji}(\gamma u)^{z_i}\Big)\rho_j(\gamma u) b_j(\gamma u).$$

Proof of Theorem 9. By Theorem 8,

$$\zeta = \sum_{j \in \mathcal{J}} \sum_{\gamma \in E_j} \zeta_{j\gamma} + \eta.$$

By Theorem 7, each f_i is bounded away from 0 on $W \setminus \widetilde{W}_0$. Additionally, because W is compact and each f_i is continuous on W, each f_i is bounded on W, and thus in particular on $W \setminus \widetilde{W}_0$. Also, $0 \leq \varphi \leq 1$ by Theorem 8. So $\sup_{w \in W \setminus \widetilde{W}_0, z \in K} |\prod_{i=1}^k f_i(w)^{z_i} \phi(w)(1 - \varphi(w))| < \infty$ for all compact $K \subseteq \mathbb{C}^k$. So by Proposition 10, η is entire.

For any $j \in \mathcal{J}$ and $\gamma \in E_j$, we show that

$$\zeta_{j\gamma}(z) := \int_{O_\gamma^\times} \prod_{l=1}^{d} |u_l|^{\sum_{i=1}^k z_i k_{jil} + h_{jl}} (\prod_{i=1}^{k} a_{ji}(u)^{z_i}) \rho_j(u) b_j(u) du$$

meromorphically continues to $\mathbb{C}^k$. Set $d' = d + k$. Set $q_l = \sum_{i=1}^k z_i k_{jil} + h_{jl}$ for $l \in [1, d]$. Set $q_l = z_{l-d}$ for $l \in [d+1, d+k]$. This means that $q = A_j(z)$ for the affine map $A_j : \mathbb{C}^k \to \mathbb{C}^{d'}$ in the statement.

Define $\widetilde{\omega}_{j\gamma} : [0,1]^d \times \mathbb{C}^{d'} \to \mathbb{C}$ by

$$\widetilde{\omega}_{j\gamma}(u, q) := \Big(\prod_{i=1}^{k} a_{ji}(\gamma u)^{q_{i+d}}\Big) \rho_j(\gamma u) b_j(\gamma u).$$

For each i, the map $(u, q) \mapsto a_{ji}(\gamma u)^{q_{i+d}}$ belongs to $A_{d'}([0,1]^d)$ by Lemma 19, because a_{ji} is a positive smooth unit. Since $A_{d'}([0,1]^d)$ is a $\mathbb{C}$-algebra by Lemma 18, and $\rho_j(\gamma u) b_j(\gamma u)$ is smooth with no q-dependence, it follows that $\widetilde{\omega}_{j\gamma} \in A_{d'}([0,1]^d)$. Thus

$$\widetilde{\zeta}_{j\gamma}(q) := \int_{[0,1]^d} \prod_{l=1}^{d} u_l^{q_l} \widetilde{\omega}_{j\gamma}(u, q) du$$

meromorphically continues to $\mathbb{C}^{d'}$. By construction, $\zeta_{j\gamma} = \widetilde{\zeta}_{j\gamma} \circ A_j$. So $\zeta_{j\gamma}$ meromorphically continues to $\mathbb{C}^k$. Applying Proposition 11 to $\widetilde{\zeta}_{j\gamma}$ gives, for every N, functions $\widetilde{\vartheta}_{j\gamma\beta} \in A_{d'}([0,1]^{B(\beta)})$. Pulling back by A_j and setting $\vartheta_{j\gamma\beta}(u, z) = \widetilde{\vartheta}_{j\gamma\beta}(u, A_j(z))$ gives the stated expansion and integrability assertion. By summing over orthants and j, ζ meromorphically continues to $\mathbb{C}^k$. Further, the polar set has the desired form because $q_l = \sum_{i=1}^k z_i k_{jil} + h_{jl}$ for all $l \in [1, d]$.

Now we compute $\zeta_{j\gamma}$ locally. If $Z = \emptyset$, we can choose $\emptyset$ as an open neighborhood of Z. Then there is no formula to show. Otherwise, $Z \neq \emptyset$. Let $\widetilde{Z} = \{q \in \mathbb{C}^{d'} : q_l + \beta_l + 1 = 0 \ \forall l \in S, \mathrm{Re}(q_l + 1) > 0 \ \forall l \in S^c\}$. By Proposition 11, there exists $\widetilde{\vartheta}_\beta : [0,1]^{S^c} \times \mathbb{C}^{d'} \to \mathbb{C}$ and $\widetilde{\xi}$ meromorphic on $\mathbb{C}^{d'}$ such that

$$\widetilde{\zeta}_{j\gamma}(q) = \frac{1}{\prod_{l \in S}(q_l + \beta_l + 1)} \Big[\int_{[0,1]^{S^c}} (\prod_{l \in S^c} u_l^{q_l}) \widetilde{\vartheta}_\beta(u_{S^c}, q) du_{S^c} \Big] + \widetilde{\xi}(q),$$

and $(\prod_{l \in S}(q_l + \beta_l + 1))\widetilde{\xi}$ vanishes on $\widetilde{Z}$. Pulling back to z coordinates, set $\vartheta_{j\gamma\beta}(u, z) := \widetilde{\vartheta}_\beta(u, A_j(z))$ and $\xi := \widetilde{\xi} \circ A_j$. Then on an open neighborhood of Z,

$$\zeta_{j\gamma}(z) = \frac{1}{\prod_{l \in S}(\sum_{i=1}^k z_i k_{jil} + h_{jl} + \beta_l + 1)} \Big[\int_{[0,1]^{S^c}} (\prod_{l \in S^c} u_l^{\sum_{i=1}^k z_i k_{jil} + h_{jl}}) \vartheta_{j\gamma\beta}(u_{S^c}, z) du_{S^c} \Big] + \xi(z),$$

and $(\prod_{l \in S}(\sum_{i=1}^k z_i k_{jil} + h_{jl} + \beta_l + 1))\xi$ vanishes on Z. Explicitly,

$$\vartheta_{j\gamma\beta} = \Big[\prod_{l \in S} \frac{\partial_{u_l}^{\beta_l}}{\beta_l!} \omega_{j\gamma}\Big]\Big|_{u_l = 0 \ \forall l \in S}, \qquad \omega_{j\gamma}(u, z) := (\prod_{i=1}^{k} a_{ji}(\gamma u)^{z_i}) \rho_j(\gamma u) b_j(\gamma u).$$

Substituting $q = A_j(z)$ and the definition of $\widetilde{\omega}_{j\gamma}(u, q)$ yields the formula for $\zeta_{j\gamma}$ stated in the theorem. $\square$

Proof of Theorem 2. Theorem 2 follows immediately from Theorem 9. □

Corollary 3. *(→ proof) Let $E \subseteq \mathbb{C}^k$ be such that Re(E) is compact in the topology of $\mathbb{R}^k$ and* $\operatorname{dist}(E, \mathcal{P}_{\mathbb{C}}) > 0$. *Then there exists $N \in \mathbb{Z}_{\geq 0}$ such that*

$$\sup_{z \in E} \frac{|\zeta(z)|}{(1 + |Im(z)|)^N} < \infty.$$

Proof of Corollary 3. By Theorem 9,

$$\zeta = \sum_{j \in \mathcal{J}} \sum_{\gamma \in E_j} \zeta_{j\gamma} + \eta.$$

Because $\operatorname{Re}(E)$ is compact and there are finitely many pairs (j, l), there is N such that

$$E \subseteq \bigcap_{j \in \mathcal{J}} \{z \in \mathbb{C}^k : \operatorname{Re}(\sum_{i=1}^{k} z_i k_{jil} + h_{jl} + N + 1) > 0 \ \forall l \in [1, d]\}.$$

If $\mathcal{J} = \emptyset$, the intersection is interpreted as $\mathbb{C}^k$. For each $j \in \mathcal{J}$ and $\gamma \in E_j$, Theorem 9 gives

$$\zeta_{j\gamma}(z) = \sum_{\beta \in \{0,\ldots,N\}^d} \frac{\int_{[0,1]^{B(\beta)}} \prod_{l \in B(\beta)} u_l^{\sum_{i=1}^{k} z_i k_{jil} + h_{jl} + N} \vartheta_{j\gamma\beta}(u_{B(\beta)}, z) du_{B(\beta)}}{\prod_{l \in B(\beta)^c} (\sum_{i=1}^{k} z_i k_{jil} + h_{jl} + \beta_l + 1)},$$

over $\{z \in \mathbb{C}^k : \operatorname{Re}(\sum_{i=1}^{k} z_i k_{jil} + h_{jl} + N + 1) > 0 \ \forall l \in [1, d]\}$. Further,

$$\int_{[0,1]^{B(\beta)}} |\prod_{l \in B(\beta)} u_l^{\sum_{i=1}^{k} z_i k_{jil} + h_{jl} + N} \vartheta_{j\gamma\beta}(u_{B(\beta)}, z)| du_{B(\beta)} < \infty,$$

over $\{z \in \mathbb{C}^k : \operatorname{Re}(\sum_{i=1}^{k} z_i k_{jil} + h_{jl} + N + 1) > 0 \ \forall l \in B(\beta)\}$. Explicitly,

$$\theta_{j\gamma\beta} := \prod_{l=1}^{d} \widetilde{S}_{l\beta_l}^{N} \omega_{j\gamma}$$
$$\theta_{j\gamma\beta}(u, z) = \vartheta_{j\gamma\beta}(u_{B(\beta)}, z)$$
$$\omega_{j\gamma}(u, z) = (\prod_{i=1}^{k} a_{ji}(\gamma u)^{z_i}) \rho_j(\gamma u) b_j(\gamma u).$$

For such j and γ, mixed partials of $\omega_{j\gamma}$ with respect to u grow at most polynomially in $|\operatorname{Im}(z)|$, so $\vartheta_{j\gamma\beta}$ grows at most polynomially in $|\operatorname{Im}(z)|$. Let

$$\widetilde{I}(z) := \int_{[0,1]^{B(\beta)}} |\prod_{l \in B(\beta)} u_l^{\sum_{i=1}^{k} z_i k_{jil} + h_{jl} + N}| du_{B(\beta)}.$$

Because $\operatorname{Re}(E)$ is compact, $\widetilde{I}$ obtains a maximum over $\operatorname{Re}(E)$. But $\widetilde{I}(z) = \widetilde{I}(\operatorname{Re}(z))$, so $\widetilde{I}$ obtains a maximum over E. So $\widetilde{I}$ is bounded over E, and the numerator integrals grow at most polynomially in $|\operatorname{Im}(z)|$. For each denominator factor

$$L_{jl\beta_l}(z) := \sum_{i=1}^{k} z_i k_{jil} + h_{jl} + \beta_l + 1,$$

either $L_{jl\beta_l}$ is a nonzero constant, or its zero set $Z_{\mathbb{C}}(L_{jl\beta_l})$ is one of the hyperplanes in $\mathcal{P}_{\mathbb{C}}$. Since $\operatorname{dist}(E, \mathcal{P}_{\mathbb{C}}) > 0$, and only finitely many such factors occur for the fixed N, these denominator factors are uniformly bounded away from 0 on E. Therefore the quotients in the expansion also grow at most polynomially in $|\operatorname{Im}(z)|$, so each $\zeta_{j\gamma}$ grows at most polynomially in $|\operatorname{Im}(z)|$ on E.

Now we consider η. By Theorem 9,

$$\eta(z) := \int_{W \setminus \widetilde{W}_0} \prod_{i=1}^{k} f_i(w)^{z_i} \phi(w)(1 - \varphi(w)) dw.$$

Then $|\eta| \leq \widetilde{\eta}$, where

$$\widetilde{\eta}(z) := \int_{W \setminus \widetilde{W}_0} |\prod_{i=1}^{k} f_i(w)^{z_i} \phi(w)(1 - \varphi(w))| dw.$$

Similarly, because $\operatorname{Re}(E)$ is compact, $\widetilde{\eta}$ obtains a maximum over $\operatorname{Re}(E)$. But $\widetilde{\eta}(z) = \widetilde{\eta}(\operatorname{Re}(z))$, so $\widetilde{\eta}$ obtains a maximum over E. So $\widetilde{\eta}$ is bounded over E, so η is bounded over E. □

Proof of Lemma 3. Let

$$O := \{z \in \mathbb{C}^k : \operatorname{Re}(\sum_{i=1}^{k} z_i k_{jil} + h_{jl} + \beta + 1) > 0 \ \forall j \in \mathcal{J}, l \in [1, d], \beta \in \mathbb{Z}_{\geq 0}\}.$$

Because $\beta \in \mathbb{Z}_{\geq 0}$,

$$O = \{z \in \mathbb{C}^k : \operatorname{Re}(\sum_{i=1}^{k} z_i k_{jil} + h_{jl} + 1) > 0 \ \forall j \in \mathcal{J}, l \in [1, d]\}.$$

Because there are finitely many inequalities, O is open. Further, $0 \in O$. Thus, there exists $c \in O \cap (-\infty, 0)^k$.

Let m_{Leb} be the Lebesgue measure on $\mathbb{R}^d$. Since μ is the push-forward of the signed measure $1_W(w)\phi(w)\, dm_{\text{Leb}}(w)$ through $(f_1, \ldots, f_k)$, its total variation satisfies

$$|\mu| \leq (f_1, \ldots, f_k)_{\#}(1_W |\phi|\, dm_{\text{Leb}}).$$

Therefore,

$$\int_{(0,\infty)^k} x^c \, d|\mu|(x) \leq \int_W \prod_{i=1}^{k} f_i(w)^{c_i} |\phi(w)|\, dm_{\text{Leb}}(w).$$

Define

$$F(w) := \prod_{i=1}^{k} f_i(w)^{c_i} |\phi(w)|.$$

Since $f_i \geq \epsilon$ on $W \setminus \widetilde{W}_0$,

$$\int_{W \setminus \widetilde{W}_0} F(w)(1 - \varphi(w))\, dm_{\text{Leb}}(w) \leq \epsilon^{\sum_{i=1}^{k} c_i} \int_{W \setminus \widetilde{W}_0} |\phi(w)|\, dm_{\text{Leb}}(w) < \infty.$$

Meanwhile, for each $j \in \mathcal{J}$ and $\gamma \in E_j$, because a_{ji} is bounded away from 0 and $\sum_{i=1}^{k} c_i k_{jil} + h_{jl} > -1$,

$$\int_{O_\gamma^\times} \prod_{l=1}^{d} |u_l|^{\sum_{i=1}^{k} c_i k_{jil} + h_{jl}} \Big(\prod_{i=1}^{k} a_{ji}(u)^{c_i}\Big) \rho_j(u) b_j(u)| \, dm_{\mathrm{Leb}}(u) < \infty.$$

For such j and γ, and for all $u \in O_\gamma^\times$, Theorem 7 gives

$$(F \circ g_j)(u)\rho_j(u)|\det Dg_j(u)| = \prod_{l=1}^{d} |u_l|^{\sum_{i=1}^{k} c_i k_{jil} + h_{jl}} \Big(\prod_{i=1}^{k} a_{ji}(u)^{c_i}\Big) \rho_j(u)|b_j(u)|.$$

$$\sum_{j \in \mathcal{J}} \sum_{\gamma \in E_j} \int_{O_\gamma^\times} (F \circ g_j)(u)\rho_j(u)|\det Dg_j(u)| \, dm_{\mathrm{Leb}}(u) + \int_{W \setminus \widetilde{W}_0} F(w)(1 - \varphi(w)) \, dm_{\mathrm{Leb}}(w) < \infty.$$

This is exactly the hypothesis in the final claim of Theorem 8. Hence $F \in L^1(W, \mathbb{C}, m_{\mathrm{Leb}})$, so

$$\int_W F(w) \, dm_{\mathrm{Leb}}(w) < \infty.$$

$$\int_{(0,\infty)^k} x^c \, d|\mu|(x) < \infty.$$

Set $c_0 := c$. □

5 Flags

In this section, we turn a flag into a coordinate system. Then, in that coordinate system, we perturb a point in the relative interior of the face at the base of the flag. We move in separated scales along different coordinates, in negative or positive directions indexed by $\alpha \in \{\pm 1\}^r$, which pick out the neighboring chambers.

5.1 Generators

The task of this subsection is to turn a flag

$$F = F_r \subsetneq \cdots \subsetneq F_0$$

into an ordered orthonormal basis for $\mathrm{lin}(F)^\perp$. To do so, we need to check that the generator defined in Proposition 3 exists and is unique.

Lemma 23. *(→ proof) Let $A \leq B$ be linear subspaces of $\mathbb{R}^k$. Suppose $\dim(B) = \dim(A) + 1$. Let $L := B \cap A^\perp$. Then $B = A \oplus L$. There exist exactly two vectors $v_1 \neq v_2 \in L$ such that $|v_1| = |v_2| = 1$. $\mathrm{span}(v_1) = \mathrm{span}(v_2) = L$. $v_1 = -v_2$.*

Proof of Lemma 23. Orthogonally decomposing B by A, $B = (A \cap B) \oplus (A^\perp \cap B) = A \oplus L$. Since $\dim(B) = \dim(A) + 1$, $\dim(L) = 1$. A 1 dimensional real inner product space has exactly two vectors with norm 1, and they are opposite each other. Any non-zero vector in a 1 dimensional vector space generates the space. □

Proof of Proposition 3. Because $\mathrm{codim}(F') = \mathrm{codim}(F) - 1$, $\dim(\mathrm{lin}(F')) = \dim(\mathrm{lin}(F)) + 1$. Let $v_1, v_2 \in \mathrm{lin}(F') \cap \mathrm{lin}(F)^\perp$ be as in Lemma 23. $\mathrm{lin}(F') = \mathrm{lin}(F) \oplus \mathrm{span}(v_1)$. Then $\mathrm{aff}(F') = \mathrm{aff}(F) + \mathbb{R}v_1$. Since F is a codimension one face of F', we also have $F' \cap \mathrm{aff}(F) = F$, and $F' \setminus F$ is convex.

There exists $z_0 \in F' \setminus F$. Since $\mathrm{aff}(F') = \mathrm{aff}(F) + \mathbb{R}v_1$, there exists $x_0 \in \mathrm{aff}(F)$ and $t_0 \in \mathbb{R}$ such that $z_0 = x_0 + t_0 v_1$. If $t_0 = 0$, then $z_0 \in F' \cap \mathrm{aff}(F) = F$, which is a contradiction, so $t_0 \neq 0$. If $t_0 > 0$, set $v = v_1$. If $t_0 < 0$, set $v = v_2$ and replace t_0 by $-t_0$. Thus $z_0 = x_0 + t_0 v$ with $x_0 \in \mathrm{aff}(F)$, $t_0 > 0$, and $z_0 \in F'$, so v satisfies condition (6). This proves existence.

1. (1) ↔ (2). Let $z \in F'$ be of the form $z = x + tu$ for $x \in \mathrm{aff}(F)$ and $t \in \mathbb{R}_{\geq 0}$. Then $z \in F$ if and only if $t = 0$.
2. (3) ↔ (4). If $x_1, x_2 \in \mathrm{aff}(F)$, $x_1 - x_2 \in \mathrm{lin}(F)$, so $\langle x_1 - x_2, u\rangle = 0$, so $\langle x, u\rangle$ is independent of $x \in \mathrm{aff}(F)$.
3. (1) → (4). Let $z \in F'$. By (1), $z = x + tu$ for some $x \in \mathrm{aff}(F)$ and $t \in \mathbb{R}_{\geq 0}$. Then $\langle z - x, u\rangle = \langle tu, u\rangle = t \geq 0$.
4. (3) → (1). Let $z \in F'$. Since $\mathrm{aff}(F') = \mathrm{aff}(F) \oplus \mathbb{R}u$, z can be uniquely written as $z = x + tu$ for some $x \in \mathrm{aff}(F)$ and $t \in \mathbb{R}$. By (3), $t = \langle tu, u\rangle = \langle z - x, u\rangle \geq 0$.
5. (2) → (6). There exists $z_0 \in F' \setminus F$. By (2), $z_0 = x_0 + t_0 u$ for some $x_0 \in \mathrm{aff}(F)$ and $t_0 \in \mathbb{R}_{>0}$, which is condition (6).

6. (6) $\to$ (1). Let $z \in F' \setminus F$. Since $\mathrm{aff}(F') = \mathrm{aff}(F) \oplus \mathbb{R}u$, z can be uniquely written as $z = x + tu$ for some $x \in \mathrm{aff}(F)$ and $t \in \mathbb{R}$. Since $z \notin F$, $t \neq 0$.

   Let $z_1, z_2 \in F' \setminus F$. Similarly, $z_1 = x_1 + t_1 u$ and $z_2 = x_2 + t_2 u$ for some $x_1, x_2 \in \mathrm{aff}(F)$ and $t_1, t_2 \in \mathbb{R}_{\neq 0}$. Suppose towards contradiction that $t_1 t_2 < 0$. Without loss of generality assume $t_2 > 0$ and $t_1 < 0$.

   Because $F' \setminus F$ and $\mathrm{aff}(F)$ are convex, for all $s \in [0,1]$, $sz_1 + (1-s)z_2 \in F' \setminus F$ and $sx_1 + (1-s)x_2 \in \mathrm{aff}(F)$. Set $s = \frac{t_2}{t_2 - t_1} \in [0,1]$, so $1 - s = \frac{-t_1}{t_2 - t_1} \in [0,1]$. Then

$$\begin{aligned} sz_1 + (1-s)z_2 &= sx_1 + st_1 u + (1-s)x_2 + (1-s)t_2 u \\ &= [sx_1 + (1-s)x_2] + [st_1 u + (1-s)t_2 u] \\ &= [sx_1 + (1-s)x_2] + [\frac{t_2 t_1}{t_2 - t_1} u + \frac{-t_1 t_2}{t_2 - t_1} u] \\ &= sx_1 + (1-s)x_2. \end{aligned}$$

   Since $sz_1 + (1-s)z_2 \in F' \setminus F$ and $sx_1 + (1-s)x_2 \in \mathrm{aff}(F)$, this contradicts $F' \cap \mathrm{aff}(F) = F$. So the sign of t is independent of z.

   By (6), there exists $x_0 \in \mathrm{aff}(F)$, $t_0 \in \mathbb{R}_{>0}$, and $z_0 \in F'$ such that $z_0 = x_0 + t_0 u$. Since $t_0 \neq 0$, we have $z_0 \notin \mathrm{aff}(F)$, hence $z_0 \notin F$. Since the sign of t is constant, t is always positive.

7. (5) $\to$ (6). There exists $x \in \mathrm{relint}(F)$. By (5), there exists $t > 0$ such that $x + tu \in \mathrm{relint}(F') \subseteq F'$. Set $z = x + tu$.

8. (1) $\to$ (5). Let $x \in \mathrm{relint}(F)$. For $t \in \mathbb{R}_{>0}$, consider $x + tu$.

   (a) Let $q \in \mathfrak{Q}(F', \mathcal{P})$. Since $\mathrm{aff}(F') = \mathrm{aff}(F) + \mathbb{R}u$, $x + tu \in \mathrm{aff}(F')$. So $q(x + tu) = 0 = \mathrm{sign}_{F'}(q)$.

   (b) Let $q \in \mathfrak{Q}(F, \mathcal{P}) \setminus \mathfrak{Q}(F', \mathcal{P})$. There exists $z_0 \in \mathrm{relint}(F')$. By (2), $z_0 = x_0 + t_0 u$ for some $x_0 \in \mathrm{aff}(F)$ and $t_0 \in \mathbb{R}_{>0}$. Let $q = l + b$ for linear l and constant b. Since $x_0 \in \mathrm{aff}(F)$ and $q \in \mathfrak{Q}(F, \mathcal{P})$, $q(x_0) = 0$, so $l(x_0) = -b$.

$$\begin{aligned} q(z_0) &= q(x_0 + t_0 u) \\ &= l(x_0 + t_0 u) + b \\ &= l(x_0) + t_0 l(u) + b \\ &= t_0 l(u). \end{aligned}$$

   Since $t_0 > 0$, $\mathrm{sign}(l(u)) = \mathrm{sign}(q(z_0)) = \mathrm{sign}_{F'}(q)$.

$$\begin{aligned} q(x + tu) &= l(x + tu) + b \\ &= l(x) + l(tu) + b \\ &= tl(u). \end{aligned}$$

   Since $t > 0$, $\mathrm{sign}(q(x + tu)) = \mathrm{sign}(l(u)) = \mathrm{sign}_{F'}(q)$.

(c) Let $q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})$. Fix a pre-compact open neighborhood U of x. Because $\mathcal{P}$ is locally finite, only finitely many hyperplanes contained in $\mathcal{P}$ meet U. For each such hyperplane H with $H \not\supseteq F$, choose a defining form $q_H \in \mathfrak{Q}(\mathcal{P})$. Since $x \in \operatorname{relint}(F)$, $q_H(x) \neq 0$. After shrinking U around x, none of these finitely many hyperplanes meets U. Thus no hyperplane defined by an element of $\mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})$ meets U. Localizing further, we can assume U is connected, so $\operatorname{sign}(q(y))$ is constant for y in this open neighborhood. So $\operatorname{sign}_{F'}(q) = \operatorname{sign}(q(x)) = \operatorname{sign}(q(x + tu))$ for small enough t.

So for small enough t, $\operatorname{sign}(q(x + tu)) = \operatorname{sign}_{F'}(q)$ for all $q \in \mathfrak{Q}(\mathcal{P})$. So $x + tu \in \operatorname{relint}(F')$.

Finally, uniqueness follows because $\operatorname{lin}(F') \cap \operatorname{lin}(F)^{\perp}$ is one-dimensional, so the only unit candidates are u and $-u$. If u satisfies condition (2), then $-u$ cannot satisfy it, since the coefficient t would have the opposite sign for the same point $z \in F' \setminus F$. Thus the generator is unique. □

Now that generators are well-defined, we define the basis associated to a flag by taking each generator of F_{i-1} over F_i, as in Definition 19.

5.2 Separated perturbations and opposite flags

Proposition 3 gives a canonical unit vector as the generator of F_{i-1} over F_i, which yields the orthonormal basis for $\operatorname{lin}(F)^{\perp}$ as in Definition 19, and correspondingly, the dual basis for $(\operatorname{lin}(F)^{\perp})^*$. Because of the gap between linear forms and affine forms, for an affine form q that vanishes on F, we cannot quite write q as a linear combination of the dual basis elements. But after subtracting constants, we can obtain a decomposition.

Definition 35 (Affine forms vanishing on a face)**.** *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement, and let F be a face of positive codimension r. Define*

$$\operatorname{Aff}_0(F) := \{\rho \in \operatorname{Aff}(\mathbb{R}^k, \mathbb{R}) : F \subseteq Z(\rho)\}.$$

Lemma 24. *(→ proof) Assume the setting of Definition 35. Let*

$$\operatorname{Ann}(\operatorname{lin}(F)) := \{\ell \in (\mathbb{R}^k)^* : \ell|_{\operatorname{lin}(F)} = 0\}.$$

For $\ell \in \operatorname{Ann}(\operatorname{lin}(F))$, let b_ℓ be the constant value of ℓ on F, and set $\rho_\ell := \ell - b_\ell$. Then the map

$$\operatorname{Ann}(\operatorname{lin}(F)) \to \operatorname{Aff}_0(F), \qquad \ell \mapsto \rho_\ell$$

is an isomorphism of real vector spaces. In particular, $\operatorname{Aff}_0(F)$ has dimension r, an ordered list $\rho_{\ell_1}, \ldots, \rho_{\ell_r}$ is a basis of $\operatorname{Aff}_0(F)$ exactly when $\ell_1, \ldots, \ell_r$ is a basis of $\operatorname{Ann}(\operatorname{lin}(F))$, and every $q \in \mathfrak{Q}(F, \mathcal{P})$ has a unique expansion in any ordered basis of $\operatorname{Aff}_0(F)$.

Proof of Lemma 24. If $\ell \in \operatorname{Ann}(\operatorname{lin}(F))$, then ℓ is constant on $\operatorname{aff}(F)$, so b_ℓ is well-defined and $\rho_\ell \in \operatorname{Aff}_0(F)$. The map $\ell \mapsto \rho_\ell$ is linear.

If $\rho_\ell = 0$, then ℓ is constant on $\mathbb{R}^k$. Since ℓ is linear, $\ell = 0$, so the map is injective. For surjectivity, let $q \in \operatorname{Aff}_0(F)$, and write $q = \ell + c$ with $\ell \in (\mathbb{R}^k)^*$ and $c \in \mathbb{R}$. If $x, y \in F$, then $0 = q(x) - q(y) = \ell(x - y)$.

Since such differences span $\mathrm{lin}(F)$, ℓ vanishes on $\mathrm{lin}(F)$. For any $y \in F$, $0 = q(y) = \ell(y) + c$, so $c = -b_\ell$ and $q = \rho_\ell$.

Finally, $\dim \mathrm{Ann}(\mathrm{lin}(F)) = \mathrm{codim}(F) = r$, and the remaining statements follow from the isomorphism and the inclusion $\mathfrak{Q}(F, \mathcal{P}) \subseteq \mathrm{Aff}_0(F)$. □

Corollary 4 (Canonical flag of faces basis)**.** *(→ proof) Assume the setting of Definition 22, denote the flag by FL, and let* gi *be the greatest-index function from Definition 23. Let $H := \mathrm{lin}(F)^\perp$, and let $u_1, \dots, u_r \in H$ be the basis determined by FL, as in Definition 19. Let $u^*_{1,H}, \dots, u^*_{r,H} \in H^*$ be the dual basis. Write $u_i^* \in (\mathbb{R}^k)^*$ for the real covector obtained from $u^*_{i,H}$ by extending it by zero on $\mathrm{lin}(F)$; equivalently,*

$$u_i^*(x_H + x_F) = u^*_{i,H}(x_H) \qquad x_H \in H,\ x_F \in \mathrm{lin}(F).$$

Then $u_i^* \in \mathrm{Ann}(\mathrm{lin}(F))$, so u_i^* is constant on F. Let b_i be this constant value and set

$$\rho_i^{FL} := u_i^* - b_i.$$

Then $\rho^{FL} := (\rho_1^{FL}, \dots, \rho_r^{FL})$ is an ordered basis of $\mathrm{Aff}_0(F)$. Moreover, if

$$q = \sum_{i=1}^{r} a_i^{FL}(q)\rho_i^{FL}$$

is the expansion of $q \in \mathfrak{Q}(F, \mathcal{P})$ in this basis, then $a_i^{FL}(q) = 0$ for all $i > \mathrm{gi}(q)$ and $a^{FL}_{\mathrm{gi}(q)}(q) \neq 0$. Equivalently,

$$q = \sum_{i=1}^{\mathrm{gi}(q)} a_i^{FL}(q)\rho_i^{FL}.$$

Thus $\mathrm{gi}(q)$ is the greatest index with nonzero coefficient in the canonical flag of faces basis.

Proof of Corollary 4. The real covectors $u_1^*, \dots, u_r^*$ form a basis of $\mathrm{Ann}(\mathrm{lin}(F))$. By Lemma 24, $\rho_1^{FL}, \dots, \rho_r^{FL}$ is an ordered basis of $\mathrm{Aff}_0(F)$.

Since $q \in P_{\mathrm{gi}(q)}$, we have $q \in \mathfrak{Q}(F_{\mathrm{gi}(q)}, \mathcal{P})$, so the linear part of q lies in

$$\mathrm{Ann}(\mathrm{lin}(F_{\mathrm{gi}(q)})) = \mathrm{span}(u_1^*, \dots, u^*_{\mathrm{gi}(q)}).$$

Therefore $a_i^{FL}(q) = 0$ for all $i > \mathrm{gi}(q)$. If $a^{FL}_{\mathrm{gi}(q)}(q) = 0$, then the expansion uses only ρ_i^{FL} with $i < \mathrm{gi}(q)$. These terms vanish on $F_{\mathrm{gi}(q)-1}$, so $q \in \mathfrak{Q}(F_{\mathrm{gi}(q)-1}, \mathcal{P})$, contradicting $q \in P_{\mathrm{gi}(q)}$. □

The decomposition from Corollary 4 will show how each $q \in \mathfrak{Q}(\mathcal{P})$ is affected as we perturb a point of F by moving in the directions $u_r, \dots, u_1$ on widely separated scales. The separation condition is designed so that for each $q \in \mathfrak{Q}(F, \mathcal{P})$, the sign of q is controlled by its greatest nonzero coordinate. This is what allows a sign vector $\alpha \in \{\pm 1\}^r$ to prescribe a neighboring chamber. Before specializing to the flag basis, we record certain notation for an arbitrary ordered basis of $\mathrm{Aff}_0(F)$, since the same notation will also be used in the residue construction.

Definition 36. *Assume the setting of Definition 35. Fix an ordered basis $\rho = (\rho_1, \dots, \rho_r)$ of $\mathrm{Aff}_0(F)$. Then every $q \in \mathfrak{Q}(F, \mathcal{P})$ has a unique expansion*

$$q = \sum_{i=1}^{r} a_i^\rho(q)\rho_i.$$

Define $\mathrm{gi}_\rho(q)$ to be the greatest index i such that $a_i^\rho(q) \neq 0$, and define $\mathrm{gc}_\rho(q) := a^\rho_{\mathrm{gi}_\rho(q)}(q)$ and

$$\mathrm{tail}_\rho(q) := q - \mathrm{gc}_\rho(q)\rho_{\mathrm{gi}_\rho(q)} = \sum_{i=1}^{\mathrm{gi}_\rho(q)-1} a_i^\rho(q)\rho_i.$$

Definition 37. *Assume the setting of Definition 36. Choose a finite set $\mathcal{Q}_F \subseteq \mathfrak{Q}(F, \mathcal{P})$ containing one representative for each hyperplane in $\{Z(q) : q \in \mathfrak{Q}(F, \mathcal{P})\}$. Choose $\tau_{0,\rho} > 0$ so large that for all $q \in \mathcal{Q}_F$,*

$$\frac{\sum_{i=1}^{\mathrm{gi}_\rho(q)-1} |a_i^\rho(q)|\tau^{i-\mathrm{gi}_\rho(q)}}{|\,\mathrm{gc}_\rho(q)|} \leq 1$$

holds whenever $\tau \geq \tau_{0,\rho}$. Since this inequality is invariant under multiplying q by a nonzero scalar, it then holds for all $q \in \mathfrak{Q}(F, \mathcal{P})$.

$\tau_{0,\rho}$ is a separation factor large enough to ensure that the greatest coordinate term dominates the tail for all $q \in \mathfrak{Q}(F, \mathcal{P})$.

For the face-flag arguments below, we specialize this notation to the canonical flag of faces basis ρ^{FL} from Corollary 4. That corollary shows that $\mathrm{gi}_{\rho^{FL}}(q)$ agrees with the combinatorial index $\mathrm{gi}(q)$ from Definition 23. After this specialization, we suppress ρ^{FL} from the notation and write ρ_i, $a_i(q)$, $\mathrm{gc}(q)$, $\mathrm{tail}(q)$, and τ_0.

Definition 38. *We say that $\epsilon \in \mathbb{R}^r_{>0}$ is τ-separated for $\tau > 0$ if for all $i \in [1, r-1]$, $\tau\epsilon_i < \epsilon_{i+1}$.*

We can now choose neighboring chambers by perturbing in the directions $u_r, \dots, u_1$ on τ_0-separated scales. The sign vector $\alpha \in \{\pm 1\}^r$ prescribes the neighboring chamber by controlling the sign of the greatest index term for each $q \in \mathfrak{Q}(F, \mathcal{P})$, which in turn controls the sign of q itself. Further, the sign vector also specifies an associated codimension i face $F_i(\alpha)$ for each $i \in [0, r]$, not just the neighboring chamber $\mathrm{relint}(F_0(\alpha))$.

Proof of Proposition 4. Fix $\alpha \in \{\pm 1\}^r$. First note the coordinate description of the original flag. For each $i \in [1, r]$, the affine forms $\rho_1, \dots, \rho_i$ form a basis of $\mathrm{Aff}_0(F_i)$. Indeed, $u_1, \dots, u_i$ is a basis for $\mathrm{lin}(F_i)^\perp$, so $u_1^*, \dots, u_i^*$ is a basis of $\mathrm{Ann}(\mathrm{lin}(F_i))$. Under the isomorphism from Lemma 24, this basis maps to $\rho_1, \dots, \rho_i$. Thus

$$\mathrm{aff}(F_i) = \bigcap_{m=1}^{i} Z(\rho_m).$$

Let $x \in \mathrm{relint}(F)$. We choose a small open neighborhood of x on which no hyperplane outside $\mathfrak{Q}(F, \mathcal{P})$ appears. More precisely, by local finiteness, after shrinking a pre-compact connected open neighborhood U of x, we may assume

$$Z(q) \cap U = \emptyset \qquad \text{for all } q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P}).$$

For each such q, $\operatorname{sign}(q)$ is constant on U. Since $x \in \overline{\mathfrak{c}}$ and U meets $\mathfrak{c}$, this constant is $\operatorname{sign}_{\mathfrak{c}}(q)$.

Choose $\epsilon \in \mathbb{R}^r_{>0}$ sufficiently small and τ_0 separated. For $i \in [0, r]$, set

$$y_i^+ := x + \sum_{j=i+1}^{r} \epsilon_j u_j, \qquad y_i(\alpha) := x + \sum_{j=i+1}^{r} \epsilon_j \alpha_j u_j.$$

We take ϵ small enough that $y_i(\alpha) \in U$ for all i, and that the all-positive points satisfy $y_i^+ \in \operatorname{relint}(F_i)$ for all i. The inclusion $y_i^+ \in \operatorname{relint}(F_i)$ follows by descending induction from $y_r^+ = x \in \operatorname{relint}(F_r)$, using

$$y_i^+ = y_{i+1}^+ + \epsilon_{i+1} u_{i+1}$$

and Proposition 3.

We now compute the signs of all arrangement forms at $y_i(\alpha)$. First, $y_i(\alpha) \in \operatorname{aff}(F_i)$. This is trivial for $i = 0$ because $\operatorname{aff}(F_0) = \mathbb{R}^k$. If $i \geq 1$ and $m \leq i$, then

$$\rho_m(y_i(\alpha)) = \rho_m(x) + \sum_{j=i+1}^{r} \epsilon_j \alpha_j u_m^*(u_j) = 0,$$

so $y_i(\alpha) \in \bigcap_{m=1}^{i} Z(\rho_m) = \operatorname{aff}(F_i)$. Hence, if $q \in \mathfrak{Q}(F_i, \mathcal{P})$, then

$$q(y_i(\alpha)) = 0.$$

Next let $q \in \mathfrak{Q}(F, \mathcal{P}) \setminus \mathfrak{Q}(F_i, \mathcal{P})$. Then $i < \operatorname{gi}(q)$ and $\rho_{\operatorname{gi}(q)}(y_i(\alpha)) = \alpha_{\operatorname{gi}(q)} \epsilon_{\operatorname{gi}(q)}$. Using

$$q = \operatorname{gc}(q) \rho_{\operatorname{gi}(q)} + \operatorname{tail}(q),$$

we get

$$q(y_i(\alpha)) = \operatorname{gc}(q) \rho_{\operatorname{gi}(q)}(y_i(\alpha)) \Big(1 + \frac{\operatorname{tail}(q)(y_i(\alpha))}{\operatorname{gc}(q) \rho_{\operatorname{gi}(q)}(y_i(\alpha))}\Big).$$

The separation condition gives

$$\Big|\frac{\operatorname{tail}(q)(y_i(\alpha))}{\operatorname{gc}(q) \rho_{\operatorname{gi}(q)}(y_i(\alpha))}\Big| \leq \frac{\sum_{m=1}^{\operatorname{gi}(q)-1} |a_m(q)| \epsilon_m}{|\operatorname{gc}(q)| \epsilon_{\operatorname{gi}(q)}} < \frac{\sum_{m=1}^{\operatorname{gi}(q)-1} |a_m(q)| \tau_0^{m - \operatorname{gi}(q)}}{|\operatorname{gc}(q)|} \leq 1.$$

Thus the last factor is positive, and

$$\operatorname{sign}(q(y_i(\alpha))) = \operatorname{sign}(\operatorname{gc}(q)) \alpha_{\operatorname{gi}(q)}.$$

Applying the same argument to y_i^+ gives $\operatorname{sign}(q(y_i^+)) = \operatorname{sign}(\operatorname{gc}(q))$. Since $y_i^+ \in \operatorname{relint}(F_i) \subseteq \overline{\mathfrak{c}}$ and $q \notin \mathfrak{Q}(F_i, \mathcal{P})$, we also have $\operatorname{sign}(q(y_i^+)) = \operatorname{sign}_{\mathfrak{c}}(q)$, so $\operatorname{sign}(\operatorname{gc}(q)) = \operatorname{sign}_{\mathfrak{c}}(q)$. Applying this back to $y_i(\alpha)$ gives

$$\operatorname{sign}(q(y_i(\alpha))) = \alpha_{\operatorname{gi}(q)} \operatorname{sign}_{\mathfrak{c}}(q).$$

Finally, if $q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})$, then $y_i(\alpha) \in U$, so

$$\operatorname{sign}(q(y_i(\alpha))) = \operatorname{sign}_{\mathfrak{c}}(q).$$

Let $F_i(\alpha)$ be the face whose relative interior contains $y_i(\alpha)$. The three sign computations give exactly the displayed sign formula. In particular, $F_i(\alpha)$ has the same vanishing forms as F_i, so $\operatorname{codim}(F_i(\alpha)) = \operatorname{codim}(F_i) = i$. If $\alpha = 1^r$, then $y_i(\alpha) = y_i^+ \in \operatorname{relint}(F_i)$, hence $F_i(1^r) = F_i$.

It remains only to check that the faces form a flag. For $i \in [1, r]$, the sign pattern of $F_i(\alpha)$ is obtained from the sign pattern of $F_{i-1}(\alpha)$ by changing exactly the forms in

$$P_i = \mathfrak{Q}(F_i, \mathcal{P}) \setminus \mathfrak{Q}(F_{i-1}, \mathcal{P})$$

from the nonzero sign $\alpha_i \operatorname{sign}_{\mathfrak{c}}(q)$ to 0. Thus $F_i(\alpha) \subseteq F_{i-1}(\alpha)$. The inclusion is proper because $P_i \neq \emptyset$. □

Proposition 4 provides a characterization of $F_i(\alpha)$ by the sign of each $q \in \mathfrak{Q}(\mathcal{P})$. We also have a characterization by identifying a specific point in the relative interior.

Corollary 5. *(→ proof) Assume the setting of Proposition 4. Let $x \in \operatorname{relint}(F)$. For all $\epsilon \in \mathbb{R}^r_{>0}$ sufficiently small and τ_0 separated, for all $i \in [0, r]$, $\operatorname{relint}(F_i(\alpha))$ contains the point $x + \sum_{j=i+1}^r \epsilon_j \alpha_j u_j$.*

Proof of Corollary 5. This is shown in the proof of Proposition 4. □

6 Multivariate residues and a jump formula

In this section, we develop a residue theory for $\widehat{R}(U, \mathcal{P})$, as in Definition 9. The residue theory is local, so we work near a fixed face F. The affine coordinate basis ρ^{FL} from Corollary 4 lets us do this directly on the affine face, without translating F to a linear subspace.

The construction below is influenced by the literature on iterated and Jeffrey-Kirwan residues [9, 32], though in the end only the iterated residue appears. Our residue Res_{FL} is equivalent to the iterated residue: the operation of applying $\mathrm{Res}_{\rho_1^{FL}=0}, \ldots, \mathrm{Res}_{\rho_r^{FL}=0}$ successively in the order determined by the flag. We instead define Res_{FL} as the coefficient $a_{-1,\ldots,-1}$ in a certain Laurent series. The τ-separated region in Definition 40 is also adapted from [32].

The main result of this section is the jump formula in Theorem 11, which expresses a signed sum of contour integrals $\int_{c(\alpha)+i\mathbb{R}^k} f\mu$, with $c(\alpha)$ ranging over different chambers, as equal to the residue integral $\int_{c+i\,\mathrm{lin}(F)} \mathrm{Res}_{FL}(f\mu)$. Also, by using our coefficient extraction formulation, we develop in Proposition 16 a formula for the residue in terms of derivatives.

6.1 Affine residues

Definition 39. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Let*

$$F = F_r \subsetneq F_{r-1} \subsetneq \cdots \subsetneq F_0$$

be a flag of faces of $\mathcal{P}$ with base F, and denote it by FL. Let $r := \mathrm{codim}(F) > 0$. Let $V \subseteq \mathbb{R}^k$ be open and set $U := V + i\mathbb{R}^k$. Assume

$$\mathrm{relint}(F) \cap V \neq \emptyset \qquad \text{and} \qquad V \cap Z(q) = \emptyset \quad \forall q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P}),$$

and assume that $U \cap Z_{\mathbb{C}}(q)$ has finitely many connected components for every $q \in \mathfrak{Q}(F, \mathcal{P})$. Thus V is localized at F: the only hyperplanes of $\mathcal{P}$ that can meet V are the hyperplanes that contain F.

Set $H := \mathrm{lin}(F)^{\perp}$, $H_{\mathbb{C}} := H + iH$, and $F_{\mathbb{C}} := \mathrm{aff}(F) + i\,\mathrm{lin}(F)$. Let $u_1, \ldots, u_r \in H$ be the basis determined by FL in Definition 19. Let $u_{1,H}^*, \ldots, u_{r,H}^* \in H^*$ be the dual basis. Write $u_i^* \in (\mathbb{R}^k)^*$ for the real covector obtained from $u_{i,H}^*$ by extending it by zero on $\mathrm{lin}(F)$, and use the same symbol for its $\mathbb{C}$-linear extension to $\mathbb{C}^k$. Let b_i be the constant value of u_i^* on F, and set

$$\rho_i := u_i^* - b_i.$$

Then $\rho = (\rho_1, \ldots, \rho_r)$ is the canonical flag of faces basis from Corollary 4. We use the same symbol ρ_i for the complex affine form $z \mapsto u_i^*(z) - b_i$. Thus $\rho_i(y) = 0$ for all $y \in F_{\mathbb{C}}$, and if $z = u + y$ with $u \in H_{\mathbb{C}}$ and $y \in F_{\mathbb{C}}$, then

$$\rho_i(z) = u_i^*(u).$$

Specialize the notation of Definitions 36 and 37 to the basis ρ, and suppress ρ from the notation. In particular, the constant τ_0 is the specialization of $\tau_{0,\rho}$ from Definition 37 to the canonical basis $\rho = \rho^{FL}$. For each $q \in \mathfrak{Q}(F, \mathcal{P})$,

$$q = \sum_{i=1}^{r} a_i(q)\rho_i.$$

Here $\mathrm{gi}(q)$ *agrees with the greatest-index function from Definition 23, by Corollary 4. In these coordinates,*

$$\mathrm{tail}(q) = q - \mathrm{gc}(q)\rho_{\mathrm{gi}(q)} = \sum_{i=1}^{\mathrm{gi}(q)-1} a_i(q)\rho_i.$$

Thus $\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}$ *is the last nonzero term of* q*, and* $\mathrm{tail}(q)$ *contains all earlier terms.*

Informally, we want to work in a region where $q \approx \mathrm{gc}(q)\rho_{\mathrm{gi}(q)}$, so that if each $\rho_i \neq 0$, then $q \neq 0$ for every $q \in \mathfrak{Q}(F, \mathcal{P})$. This inspires Definition 40. The inequalities in $U(\tau)$ impose a hierarchy of scales among the transverse coordinates: each coordinate is much larger than the ones that come before it. This is precisely the regime in which the leading term $\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}$ dominates the lower-order terms in q. Throughout this subsection, we regularly package the lower-order terms into $\mathrm{tail}(q)$ as defined above, and use the estimate from Lemma 25 to show that the tail is dominated by the leading term.

Definition 40. *Assume the setting of Definition 39. For* $\tau > 0$*, define*

$$U(\tau) := \{z = u + y : u \in H_{\mathbb{C}}, y \in F_{\mathbb{C}}, 0 < |\rho_1(z)|, \tau|\rho_i(z)| < |\rho_{i+1}(z)| \ \forall i \in [1, r-1]\}.$$

Lemma 25. *(→ proof) Assume the setting of Definition 39. Let* $\tau \geq \tau_0$ *and let* $U(\tau)$ *be as in Definition 40. For all* $q \in \mathfrak{Q}(F, \mathcal{P})$ *and* $z \in U(\tau)$*,*

$$\Big|\frac{\mathrm{tail}(q)(z)}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}(z)}\Big| < \frac{\tau_0}{\tau} \leq 1.$$

Proof of Lemma 25. Let $z \in U(\tau)$.

$$\begin{aligned}\Big|\frac{\mathrm{tail}(q)(z)}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}(z)}\Big| &= \Big|\frac{\sum_{i=1}^{\mathrm{gi}(q)-1} a_i(q)\rho_i(z)}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}(z)}\Big| \\ &= \Big|\sum_{i=1}^{\mathrm{gi}(q)-1} \frac{a_i(q)\rho_i(z)}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}(z)}\Big|\end{aligned}$$

Because $z \in U(\tau)$, if $i < \mathrm{gi}(q)$, $|\frac{\rho_i(z)}{\rho_{\mathrm{gi}(q)}(z)}| < \tau^{i-\mathrm{gi}(q)} \leq \tau_0^{i-\mathrm{gi}(q)}\frac{\tau_0}{\tau}$. Then

$$\Big|\sum_{i=1}^{\mathrm{gi}(q)-1} \frac{a_i(q)\rho_i(z)}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}(z)}\Big| \leq \sum_{i=1}^{\mathrm{gi}(q)-1} \Big|\frac{a_i(q)\rho_i(z)}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}(z)}\Big| < \sum_{i=1}^{\mathrm{gi}(q)-1} \Big|\frac{a_i(q)}{\mathrm{gc}(q)}\Big|\tau_0^{i-\mathrm{gi}(q)}\frac{\tau_0}{\tau} \leq \frac{\tau_0}{\tau}$$

□

Separated scales can also easily be represented by a torus, as in the following definition.

Definition 41. *Assume the setting of Definition 39. For* $\epsilon \in \mathbb{R}^r_{>0}$ *and* $y \in F_{\mathbb{C}}$*, we define the torus*

$$T_\epsilon(y) = \{z = u + y : u \in H_{\mathbb{C}}, |\rho_i(z)| = \epsilon_i \ \forall i \in [1, r]\},$$

oriented by $\bigwedge_{i=1}^r d\arg\rho_i$.

The torus varies only in the transverse directions and keeps the point y on $F_{\mathbb{C}}$ fixed.

Lemma 26. *(→ proof) Let $y \in F_{\mathbb{C}}$. $\epsilon \in \mathbb{R}^r_{>0}$ is τ-separated if and only if $T_\epsilon(y) \subseteq U(\tau)$.*

Proof of Lemma 26. This is a result of unpacking definitions. □

The next proposition packages the consequences of the separation condition for defining the residue.

Proposition 12. *(→ proof) Assume the setting of Definition 39. Let $\tau \geq \tau_0$.*

1. *$U(\tau) \cap U \subseteq \mathbb{C}^k \setminus \mathcal{P}_{\mathbb{C}}$.*
2. *For all $q \in \mathfrak{Q}(F, \mathcal{P})$, $\frac{1}{q}$ admits a Laurent series expansion in $\rho_1, \ldots, \rho_r$ over $U(\tau)$.*
3. *Let $f \in \widehat{R}(U, \mathcal{P})$. f admits a Laurent series expansion in $\rho_1, \ldots, \rho_r$ over $U(\tau) \cap U$.*

Proof of Proposition 12. For (1), let $z \in U(\tau) \cap U$. If $q \in \mathfrak{Q}(F, \mathcal{P})$, then

$$q = \mathrm{gc}(q)\rho_{\mathrm{gi}(q)}(1 + \frac{\mathrm{tail}(q)}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}}).$$

$\mathrm{gc}(q) \neq 0$ by definition, $\rho_{\mathrm{gi}(q)}(z) \neq 0$ by definition of $U(\tau)$, and the last factor is nonzero by Lemma 25. So $q(z) \neq 0$. If $q \notin \mathfrak{Q}(F, \mathcal{P})$, then $\mathrm{Re}(z) \in V$ and Definition 39 assumes $V \cap Z(q) = \emptyset$. Hence $q(\mathrm{Re}(z)) \neq 0$, and therefore $q(z) = q(\mathrm{Re}(z)) + iq(\mathrm{Im}(z)) \neq 0$. Thus $z \notin \mathcal{P}_{\mathbb{C}}$.

For (2), we show that q admits a Laurent series expansion in the transverse variables over $U(\tau)$.

$$\begin{aligned}
\frac{1}{q} &= \frac{1}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)} + \mathrm{tail}(q)} \\
&= \frac{1}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}} \frac{1}{1 + \frac{\mathrm{tail}(q)}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}}} \\
&= \frac{1}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}} \sum_{j=0}^{\infty} (-1)^j (\frac{\mathrm{tail}(q)}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}})^j.
\end{aligned}$$

The series converges because $|\frac{\mathrm{tail}(q)}{\mathrm{gc}(q)\rho_{\mathrm{gi}(q)}}| < 1$ over $U(\tau)$. This proves (2). For (3), Lemma 8 gives $\widehat{R}(U, \mathcal{P}) = R(U, \mathcal{P})$, and any denominator that can meet U comes from $\mathfrak{Q}(F, \mathcal{P})$. Thus the claim follows from (2). □

At this point we can treat functions in $\widehat{R}(U, \mathcal{P})$ as Laurent series in the variables $\rho_1, \ldots, \rho_r$, with coefficients that are holomorphic along $F_{\mathbb{C}} \cap U$.

Definition 42 (Residue)**.** *Assume the setting of Definition 39. Let $f \in \widehat{R}(U, \mathcal{P})$. For $\beta \in \mathbb{Z}^r$, define $\rho^\beta = \prod_{i=1}^r \rho_i^{\beta_i}$. More generally, if W is open in the subspace topology of an affine subspace $A \subseteq \mathbb{C}^k$, let $\mathcal{O}(W)$ refer to the space of holomorphic functions over W as an open neighborhood of A. Let $f = \sum_{\beta \in \mathbb{Z}^r} a_\beta \rho^\beta$ be the Laurent series of f over $U(\tau_0) \cap U$, with $a_\beta \in \mathcal{O}(F_{\mathbb{C}} \cap U)$.*

Let

$$\eta_H := d\rho_1 \wedge \cdots \wedge d\rho_r \in \Lambda^r H^*_{\mathbb{C}}.$$

Since $\eta_H \neq 0$ and $\mathbb{C}^k = H_{\mathbb{C}} \oplus \operatorname{lin}(F)_{\mathbb{C}}$, the map $\Lambda^{k-r} \operatorname{lin}(F)^*_{\mathbb{C}} \to \Lambda^k(\mathbb{C}^k)^*$ given by $\lambda_F \mapsto \eta_H \wedge \lambda_F$ is an isomorphism of one-dimensional vector spaces. Thus every $\mu \in \Lambda^k(\mathbb{C}^k)^*$ is written uniquely as $\mu = \eta_H \wedge \lambda_F$.

Finally, define

$$\operatorname{Res}_{FL} : \widehat{R}(U, \mathcal{P}) \otimes \Lambda^k(\mathbb{C}^k)^* \to \mathcal{O}(F_{\mathbb{C}} \cap U) \otimes \Lambda^{k-r} \operatorname{lin}(F)^*_{\mathbb{C}}$$

by

$$\operatorname{Res}_{FL}(f\mu) := a_{-1,\dots,-1}\lambda_F.$$

In other words, we freeze the tangential variable and extract the coefficient of $\prod_{i=1}^r \rho_i^{-1}$ in the Laurent expansion in the transverse variables. This is a multivariate analogue of taking the coefficient of u^{-1} in one variable.

The next theorem shows that this coefficient-extraction definition agrees with a contour integral over a small torus. This equivalence is critical to later make the residue compatible with vertical contour integrals.

Theorem 10. *(→ proof) Assume the setting of Definition 39. Let $f \in \widehat{R}(U, \mathcal{P})$. Let $\tau \geq \tau_0$. Then for all $y \in F_{\mathbb{C}} \cap U$, there is $\delta > 0$ such that for all $\epsilon \in \mathbb{R}^r_{>0}$ that is τ-separated and $|\epsilon| < \delta$,*

$$\operatorname{Res}_{FL}(f\,\eta_H \wedge \lambda_F)(y) = [\frac{1}{(2\pi i)^r} \int_{T_\epsilon(y)} f\,\eta_H]\lambda_F$$

for all $\lambda_F \in \Lambda^{k-r} \operatorname{lin}(F)^*_{\mathbb{C}}$.

Proof of Theorem 10. Because $\tau \geq \tau_0$, f locally admits an expansion

$$f = \sum_{\beta \in \mathbb{Z}^r} a_\beta \rho^\beta = a_{-1,\dots,-1} \prod_{i=1}^r \rho_i^{-1} + \sum_{\beta \in \mathbb{Z}^r \setminus \{-1\}^r} a_\beta \rho^\beta$$

over $U(\tau) \cap U$. The second term is exact over the slice of $U(\tau) \cap U$ with y fixed. Because ϵ is τ separated and sufficiently small, $T_\epsilon(y) \subseteq U(\tau) \cap U$. Thus

$$\begin{aligned}
[\frac{1}{(2\pi i)^r} \int_{T_\epsilon(y)} f\,\eta_H]\lambda_F &= [\frac{1}{(2\pi i)^r} \int_{\prod_{i=1}^r \{|\rho_i| = \epsilon_i\}} a_{-1,\dots,-1}(y) \prod_{i=1}^r \rho_i^{-1} \eta_H]\lambda_F \\
&= a_{-1,\dots,-1}(y)\lambda_F \\
&= \operatorname{Res}_{FL}(f\,\eta_H \wedge \lambda_F)(y).
\end{aligned}$$

□

6.2 Jump formula

We now turn the residue construction into the jump formula. The main geometric idea is the following. By Theorem 10, around a fixed face F, the residue Res_{FL} admits a formula as an integral over small tori in transverse directions. But for our contour shifting problem, we want integrals over vertical contours $c + i\mathbb{R}^k$. So we replace those tori by large rectangular contours whose sides are parallel to the real and imaginary axes.

The boundary of the rectangles decomposes into faces. The vertical faces, indexed by $\alpha \in \{\pm 1\}^r$, are the pieces that survive in the jump formula and correspond to approaching F from the chambers adjacent to it. The remaining faces have at least one imaginary side, and for the argument to work we need their contributions to vanish as the rectangles are pushed to infinity. With the function space $\Theta_{FL}(U, \mathcal{P})$ in Definition 44, we require sufficient decay of the function f in the imaginary directions so that the contributions from the non-vertical faces vanish in the limit. The jump formula then states that the signed sum of the contributions from the vertical contours in each chamber is equal to the residue integral along F.

The content of this subsection is to make the above argument precise by defining the rectangular contours, relating the torus integrals to the rectangular contour integrals, and defining the function space $\Theta_{FL}(U, \mathcal{P})$ to ensure the necessary decay conditions to control the contribution of the non-vertical faces.

Definition 43. *Assume the setting of Definition 39. Let $\Phi : H_{\mathbb{C}} \to \mathbb{C}^r$ be the isomorphism*

$$\Phi(u) := (u_1^*(u), \ldots, u_r^*(u)).$$

For $\epsilon, s \in \mathbb{R}_{>0}$, let $\gamma_{\epsilon,s} \subsetneq \mathbb{C}$ be the boundary of the rectangle in one variable with corners $\pm\epsilon \pm s\epsilon i$, with counter-clockwise orientation. For $\alpha \in \{\pm 1, \pm i\}$, denote the side

$$\gamma_{\epsilon,s}^{\alpha} := \begin{cases} \{\alpha\epsilon + it : t \in [-s\epsilon, s\epsilon]\} & : \alpha = \pm 1 \\ \{t + \alpha s\epsilon : t \in [-\epsilon, \epsilon]\} & : \alpha = \pm i \end{cases},$$

oriented consistently with $\gamma_{\epsilon,s}$. Next for $\epsilon \in \mathbb{R}_{>0}^r$, $s \in \mathbb{R}_{>0}$, and $y \in \mathbb{C}^k$, define a contour in $\mathbb{C}^k$ by

$$\Gamma_{\epsilon,s}(y) := \Phi^{-1}(\gamma_{\epsilon_1,s} \times \cdots \times \gamma_{\epsilon_r,s}) + y,$$

and for $\alpha \in \{\pm 1, \pm i\}^r$ define

$$\Gamma_{\epsilon,s}^{\alpha}(y) := \Phi^{-1}(\gamma_{\epsilon_1,s}^{\alpha_1} \times \cdots \times \gamma_{\epsilon_r,s}^{\alpha_r}) + y.$$

Note that $\sum_{\alpha \in \{\pm 1, \pm i\}^r} \Gamma_{\epsilon,s}^{\alpha}(y) = \Gamma_{\epsilon,s}(y)$ as r-chains. Finally, for $\alpha \in \{\pm 1, \pm i\}^r$, define

$$\Gamma_{\epsilon,\infty}^{\alpha}(y) := \bigcup_{s \geq 1} \Gamma_{\epsilon,s}^{\alpha}(y).$$

For $\alpha \in \{\pm 1\}^r$, this is the vertical contour

$$\{u + y : u \in H_{\mathbb{C}}, Re(u_i^*(u)) = \alpha_i \epsilon_i \ \forall i \in [1, r]\},$$

oriented consistently with $\Gamma_{\epsilon,s}^{\alpha}(y)$ for any $s \in \mathbb{R}_{>0}$.

The chains $\Gamma_{\epsilon,s}^{\alpha}(y)$ are the individual faces of the product rectangle. The limiting contours $\Gamma_{\epsilon,\infty}^{\alpha}(y)$ help us recover vertical contours of the form $c + i\mathbb{R}^k$ after integrating over the imaginary part of y.

The next lemma is a rectangular extension of Lemma 25: on each face of the rectangle, the leading coordinate of q still dominates, so no hyperplane equation can vanish there. This will be critical to show that the torus and rectangular contours are homotopic in the complement of the arrangement, so that the residue can be computed by integrating over the rectangular contours.

Lemma 27. *(→ proof) Assume the setting of Definition 39. Suppose $\tau \geq \tau_0$ and $\epsilon \in \mathbb{R}^r_{>0}$ is τ separated. Let $q \in \mathfrak{Q}(F, \mathcal{P})$. Let $u \in H_{\mathbb{C}}$. Suppose either*

1. *$|Re(u^*_{\mathrm{gi}(q)}(u))| = \epsilon_{\mathrm{gi}(q)}$ and $|Re(u^*_i(u))| \leq \epsilon_i$ for all $i < \mathrm{gi}(q)$.*
2. *$|Im(u^*_{\mathrm{gi}(q)}(u))| = \epsilon_{\mathrm{gi}(q)}$ and $|Im(u^*_i(u))| \leq \epsilon_i$ for all $i < \mathrm{gi}(q)$.*

Then $q(u+y) \neq 0$ for every $y \in F_{\mathbb{C}}$. In the first case, $\mathrm{sign}(Re(q(u+y))) = \mathrm{sign}(\mathrm{gc}(q) Re(u^*_{\mathrm{gi}(q)}(u)))$. In the second case, $\mathrm{sign}(Im(q(u+y))) = \mathrm{sign}(\mathrm{gc}(q) Im(u^*_{\mathrm{gi}(q)}(u)))$. Moreover, $|q(u+y)| > |\,\mathrm{gc}(q)| \epsilon_{\mathrm{gi}(q)} (1 - \frac{\tau_0}{\tau})$.

Proof of Lemma 27. We show the first case. The argument for the second case is similar. Since $\rho_i(u+y) = u^*_i(u)$, we have $q(u+y) = \sum_{i=1}^r a_i(q) u^*_i(u)$. Then

$$\begin{aligned}
|\frac{\mathrm{Re}(\mathrm{tail}(q)(u+y))}{\mathrm{Re}(\mathrm{gc}(q) u^*_{\mathrm{gi}(q)}(u))}| &= |\frac{\sum_{i=1}^{\mathrm{gi}(q)-1} a_i(q) \mathrm{Re}(u^*_i(u))}{\mathrm{gc}(q) \mathrm{Re}(u^*_{\mathrm{gi}(q)}(u))}| \\
&\leq \frac{\sum_{i=1}^{\mathrm{gi}(q)-1} |a_i(q)| \epsilon_i}{|\,\mathrm{gc}(q)| \epsilon_{\mathrm{gi}(q)}} \\
&< \frac{\sum_{i=1}^{\mathrm{gi}(q)-1} |a_i(q)| \tau^{i-\mathrm{gi}(q)}}{|\,\mathrm{gc}(q)|} \\
&\leq \frac{\tau_0}{\tau} \\
&\leq 1
\end{aligned}$$

We also have

$$\mathrm{Re}(q(u+y)) = \mathrm{Re}(\mathrm{gc}(q) u^*_{\mathrm{gi}(q)}(u))(1 + \frac{\mathrm{Re}(\mathrm{tail}(q)(u+y))}{\mathrm{Re}(\mathrm{gc}(q) u^*_{\mathrm{gi}(q)}(u))}).$$

No factor is 0, and the last factor is positive, so the sign is determined by the first factor. □

Proposition 13. *(→ proof) Assume the setting of Definition 39. Let $y \in F_{\mathbb{C}} \cap U$. Suppose $\tau \geq \tau_0$ and $\epsilon \in \mathbb{R}^r_{>0}$ is sufficiently small and τ separated. Then*

$$\Gamma_{\epsilon,s}(y) \subsetneq U \cap (\mathbb{C}^k \setminus \mathcal{P}_{\mathbb{C}})$$

for all $s \in \mathbb{R}_{>0}$. Further, if $\tau \geq \sqrt{2}\tau_0$, then $\Gamma_{\epsilon,s}(y)$ is homotopic to $T_\epsilon(y)$ in $U \cap (\mathbb{C}^k \setminus \mathcal{P}_{\mathbb{C}})$.

Proof of Proposition 13. Because V is open and $\mathrm{Re}(y) \in V$, for ϵ sufficiently small the real parts of all points in $\Gamma_{\epsilon,\infty}(y)$ lie in V. Hence the contours lie in U. Notice that $s\epsilon$ is τ separated.

We first show that no defining affine form vanishes on $\Gamma_{\epsilon,s}(y)$. Let $q \in \mathfrak{Q}(\mathcal{P})$. If $q \in \mathfrak{Q}(F, \mathcal{P})$, then

$$\Gamma_{\epsilon,s}(y) = \bigcup_{\alpha \in \{\pm 1, \pm i\}^r} \Gamma^\alpha_{\epsilon,s}(y).$$

For each $\alpha \in \{\pm 1, \pm i\}^r$, whether $\alpha_{\mathrm{gi}(q)} = \pm 1$ or $\alpha_{\mathrm{gi}(q)} = \pm i$, one of the two cases of Lemma 27 applies. Thus $q \neq 0$ anywhere on $\Gamma^\alpha_{\epsilon,s}(y)$, and hence $q \neq 0$ anywhere on $\Gamma_{\epsilon,s}(y)$.

If $q \notin \mathfrak{Q}(F, \mathcal{P})$, then every point $z \in \Gamma_{\epsilon,s}(y)$ has $\mathrm{Re}(z) \in V$, and Definition 39 gives $V \cap Z(q) = \emptyset$. Thus $q(\mathrm{Re}(z)) \neq 0$. Since q is real affine, $\mathrm{Re}(q(z)) = q(\mathrm{Re}(z))$, so $q(z) \neq 0$. Therefore $q \neq 0$ anywhere on $\Gamma_{\epsilon,s}(y)$ in this case as well.

Since every defining form is nonzero on $\Gamma_{\epsilon,s}(y)$, we have

$$\Gamma_{\epsilon,s}(y) \subsetneq U \cap (\mathbb{C}^k \setminus \mathcal{P}_{\mathbb{C}}).$$

The contours $\Gamma_{\epsilon,1}(y)$ and $\Gamma_{\epsilon,s}(y)$ are homotopic in $U \cap (\mathbb{C}^k \setminus \mathcal{P}_{\mathbb{C}})$ by a straight path. It remains to show that $\Gamma_{\epsilon,1}(y)$ is homotopic to $T_\epsilon(y)$.

Consider the continuous path π_i from $\gamma_{\epsilon_i,1}$ to the circle $\{z \in \mathbb{C} : |z| = \epsilon_i\}$ given by

$$\pi_i(t, x) = \frac{x}{(1-t) + t\frac{|x|}{\epsilon_i}}.$$

Write $\pi_i(t, x) = \lambda_i(t, x)x$. Since $1 \leq (1-t) + t\frac{|x|}{\epsilon_i} \leq \sqrt{2}$ on $\gamma_{\epsilon_i,1}$, we have $\lambda_i(t, x) \in [1/\sqrt{2}, 1]$. Thus π_i preserves the signs of the nonzero real and imaginary coordinates, and

$$|\mathrm{Re}(\pi_i(t, x))| = \lambda_i(t, x)|\mathrm{Re}(x)|, \qquad |\mathrm{Im}(\pi_i(t, x))| = \lambda_i(t, x)|\mathrm{Im}(x)|.$$

Let π be the product path from

$$\gamma_{\epsilon_1,1} \times \cdots \times \gamma_{\epsilon_r,1} \quad \text{to} \quad \{|z_1| = \epsilon_1\} \times \cdots \times \{|z_r| = \epsilon_r\}.$$

Let $u + y \in \Gamma^{\alpha}_{\epsilon,1}(y)$ and let $t \in [0, 1]$. Let $q \in \mathfrak{Q}(F, \mathcal{P})$, put $w_i = \Phi(u)_i$, and write $\ell_i = \lambda_i(t, w_i)$.

Set $\epsilon'_i = \ell_i \epsilon_i$. Because ϵ is $\sqrt{2}\tau_0$ separated and $\ell_i \in [1/\sqrt{2}, 1]$, ϵ' is τ_0 separated. If $\alpha_{\mathrm{gi}(q)} = \pm 1$, then

$$|\mathrm{Re}(\pi_{\mathrm{gi}(q)}(t, w_{\mathrm{gi}(q)}))| = \epsilon'_{\mathrm{gi}(q)}, \qquad |\mathrm{Re}(\pi_i(t, w_i))| \leq \epsilon'_i \quad i < \mathrm{gi}(q).$$

By Lemma 27, $q(\Phi^{-1}(\pi(t, \Phi(u))) + y) \neq 0$. If $\alpha_{\mathrm{gi}(q)} = \pm i$, then

$$|\mathrm{Im}(\pi_{\mathrm{gi}(q)}(t, w_{\mathrm{gi}(q)}))| = \epsilon'_{\mathrm{gi}(q)}, \qquad |\mathrm{Im}(\pi_i(t, w_i))| \leq \epsilon'_i \quad i < \mathrm{gi}(q),$$

so Lemma 27 again gives $q(\Phi^{-1}(\pi(t, \Phi(u))) + y) \neq 0$.

For $q \notin \mathfrak{Q}(F, \mathcal{P})$, the assumption $V \cap Z(q) = \emptyset$ in Definition 39 and the fact that $\Gamma_{\epsilon,s}(y) \subsetneq U = V + i\mathbb{R}^k$ excludes vanishing.

Pulling back to $\mathbb{C}^k$, $\Gamma_{\epsilon,1}(y)$ is homotopic to $T_\epsilon(y)$ in $U \cap (\mathbb{C}^k \setminus \mathcal{P}_{\mathbb{C}})$. □

This proposition lets us connect the residue to the rectangular contour integrals. To send the rectangles to infinity, we need to control the contribution of the non-vertical faces. This is the role of the function space $\Theta_{FL}(U, \mathcal{P})$.

Definition 44. *Assume the setting of Definition 39. Define $\Theta_{FL}(U, \mathcal{P}) \subseteq \widehat{R}(U, \mathcal{P})$ to be the vector space of functions $f \in \widehat{R}(U, \mathcal{P})$ such that*

1. *for the rectangular contours* $\Gamma^{\alpha}_{\epsilon,s}(y)$ *as in Definition 43,*

$$\lim_{s\to\infty}\int_{\Gamma^{\alpha}_{\epsilon,s}(y)} f\nu = 0$$

for all $\alpha \in \{\pm 1, \pm i\}^r \setminus \{\pm 1\}^r$, $\epsilon \in \mathbb{R}^r_{>0}$ *sufficiently small and separated,* $y \in (\mathrm{relint}(F) + i\,\mathrm{lin}(F)) \cap U$ *and* $\nu \in \Lambda^r H^*_{\mathbb{C}}$,

2. $\int_{c+i\mathbb{R}^k} |f\mu| < \infty$ *for all* $c \in V \setminus \mathcal{P}$ *and* $\mu \in \Lambda^k(\mathbb{C}^k)^*$,

3. $\int_{c+i\mathbb{R}^k} f\mu$ *is locally constant with respect to* $c \in V \setminus \mathcal{P}$ *for all* $\mu \in \Lambda^k(\mathbb{C}^k)^*$.

Definition 45. *For* $\alpha \in \{\pm 1\}^r$, *define* $\mathrm{sign}(\alpha) = \prod_{i=1}^r \alpha_i$.

We now state the main theorem of this section.

Theorem 11 (Jump formula)**.** *(→ proof) Assume the setting of Definition 39. Let* $f \in \Theta_{FL}(U, \mathcal{P})$ *and let* $\mu \in \Lambda^k(\mathbb{C}^k)^*$. *Let* $\mathfrak{b}_{FL}(\alpha)$ *be as in Proposition 4. Then for any* $c(\alpha) \in \mathfrak{b}_{FL}(\alpha) \cap V$ *and* $c \in \mathrm{relint}(F) \cap V$,

$$\sum_{\alpha\in\{\pm 1\}^r} \frac{\mathrm{sign}(\alpha)}{(2\pi i)^r}\int_{c(\alpha)+i\mathbb{R}^k} f\mu = \int_{c+i\,\mathrm{lin}(F)} \mathrm{Res}_{FL}(f\mu)$$

Proof of Theorem 11. The proof has four steps: express the residue as a torus integral, replace the torus by a rectangle, let the rectangle tend to infinity, and identify the surviving vertical faces with the chamber integrals appearing in the statement. Let $\mu = \eta_H \wedge \lambda_F$ be as in Definition 42, with $\lambda_F \in \Lambda^{k-r}\,\mathrm{lin}(F)^*_{\mathbb{C}}$. Let $\epsilon \in \mathbb{R}^r_{>0}$ be sufficiently small and $\sqrt{2}\tau_0$-separated, where τ_0 is the separation constant for the flag coordinates. Take the decomposition $\mathbb{C}^k = H_{\mathbb{C}} \oplus \mathrm{lin}(F)_{\mathbb{C}}$ along $F_{\mathbb{C}}$. Fix $y_0 \in \mathrm{relint}(F) \cap V$ and let $t \in \mathrm{lin}(F)$ vary. By Theorem 10,

$$\mathrm{Res}_{FL}(f\,\eta_H \wedge \lambda_F)(y_0 + it) = [\frac{1}{(2\pi i)^r}\int_{T_\epsilon(y_0+it)} f\,\eta_H]\lambda_F.$$

By Proposition 13, for all $s > 0$,

$$[\frac{1}{(2\pi i)^r}\int_{T_\epsilon(y_0+it)} f\,\eta_H]\lambda_F = [\frac{1}{(2\pi i)^r}\int_{\Gamma_{\epsilon,s}(y_0+it)} f\,\eta_H]\lambda_F.$$

Decompose $\Gamma_{\epsilon,s}(y_0 + it) = \sum_{\alpha\in\{\pm 1,\pm i\}^r} \Gamma^{\alpha}_{\epsilon,s}(y_0 + it)$ to get

$$[\frac{1}{(2\pi i)^r}\int_{\Gamma_{\epsilon,s}(y_0+it)} f\,\eta_H]\lambda_F = [\sum_{\alpha\in\{\pm 1,\pm i\}^r} \frac{1}{(2\pi i)^r}\int_{\Gamma^{\alpha}_{\epsilon,s}(y_0+it)} f\,\eta_H]\lambda_F.$$

We study these terms. First, let $\alpha \notin \{\pm 1\}^r$. Then by condition (1) defining $\Theta_{FL}(U, \mathcal{P})$,

$$\lim_{s\to\infty}\int_{\Gamma^{\alpha}_{\epsilon,s}(y_0+it)} f\,\eta_H = 0.$$

On the other hand, let $\alpha \in \{\pm 1\}^r$. Let $u_1, \ldots, u_r \in H$ be the dual basis of $u_1^*, \ldots, u_r^*$. Define $x_\alpha := \sum_{i=1}^r \alpha_i\epsilon_i u_i$. Because ϵ is sufficiently small and separated, the real point $x_\alpha + y_0$ lies in $V \setminus \mathcal{P}$ by Proposition 12. By condition (2) defining $\Theta_{FL}(U, \mathcal{P})$,

$$\int_{x_\alpha+y_0+i\mathbb{R}^k} |f\,\eta_H \wedge \lambda_F| < \infty.$$

Taking slices, for almost all $t \in \operatorname{lin}(F)$,

$$\int_{x_\alpha + iH + y_0 + it} |f\,\eta_H| < \infty.$$

$\Gamma^\alpha_{\epsilon,\infty}(y_0 + it) = x_\alpha + iH + y_0 + it$ as sets. With H oriented by $u_1^*, \ldots, u_r^*$, the orientation matches exactly when $\operatorname{sign}(\alpha) = 1$. Thus

$$\int_{\Gamma^\alpha_{\epsilon,\infty}(y_0+it)} f\,\eta_H = \operatorname{sign}(\alpha) \int_{x_\alpha + iH + y_0 + it} f\,\eta_H,$$

which we have shown is finite for almost all t. For such t, using $|f|$ as the dominating function for the dominated convergence theorem,

$$\lim_{s\to\infty} \int_{\Gamma^\alpha_{\epsilon,s}(y_0+it)} f\,\eta_H = \int_{\Gamma^\alpha_{\epsilon,\infty}(y_0+it)} f\,\eta_H = \operatorname{sign}(\alpha) \int_{x_\alpha + iH + y_0 + it} f\,\eta_H.$$

Combining the results so far, for almost all t,

$$\begin{aligned}
&\operatorname{Res}_{FL}(f\,\eta_H \wedge \lambda_F)(y_0 + it) \\
&= [\frac{1}{(2\pi i)^r} \int_{T_\epsilon(y_0+it)} f\,\eta_H]\lambda_F \\
&= [\frac{1}{(2\pi i)^r} \int_{\Gamma_{\epsilon,s}(y_0+it)} f\,\eta_H]\lambda_F \\
&= [\sum_{\alpha\in\{\pm 1,\pm i\}^r} \frac{1}{(2\pi i)^r} \int_{\Gamma^\alpha_{\epsilon,s}(y_0+it)} f\,\eta_H]\lambda_F \\
&= [\lim_{s\to\infty} \sum_{\alpha\in\{\pm 1,\pm i\}^r} \frac{1}{(2\pi i)^r} \int_{\Gamma^\alpha_{\epsilon,s}(y_0+it)} f\,\eta_H]\lambda_F \\
&= [\lim_{s\to\infty} \sum_{\alpha\in\{\pm 1\}^r} \frac{1}{(2\pi i)^r} \int_{\Gamma^\alpha_{\epsilon,s}(y_0+it)} f\,\eta_H + \frac{1}{(2\pi i)^r} \sum_{\alpha\in\{\pm 1,\pm i\}^r\setminus\{\pm 1\}^r} \int_{\Gamma^\alpha_{\epsilon,s}(y_0+it)} f\,\eta_H]\lambda_F \\
&= [\sum_{\alpha\in\{\pm 1\}^r} \frac{\operatorname{sign}(\alpha)}{(2\pi i)^r} \int_{x_\alpha + iH + y_0 + it} f\,\eta_H]\lambda_F.
\end{aligned}$$

Condition (2) defining $\Theta_{FL}(U, \mathcal{P})$ justifies Fubini's theorem to integrate out t, so

$$\begin{aligned}
\int_{y_0 + i\operatorname{lin}(F)} \operatorname{Res}_{FL}(f\,\eta_H \wedge \lambda_F) &= [\sum_{\alpha\in\{\pm 1\}^r} \frac{\operatorname{sign}(\alpha)}{(2\pi i)^r} \int_{x_\alpha + iH + y_0 + i\operatorname{lin}(F)} f\,\eta_H]\lambda_F \\
&= \sum_{\alpha\in\{\pm 1\}^r} \frac{\operatorname{sign}(\alpha)}{(2\pi i)^r} \int_{x_\alpha + y_0 + i\mathbb{R}^k} f\,\eta_H \wedge \lambda_F.
\end{aligned}$$

By condition (3) defining $\Theta_{FL}(U, \mathcal{P})$, each term on the right-hand side is locally constant as the real base point varies in $V \setminus \mathcal{P}$. Thus the left-hand side is also independent of the base point in $\operatorname{relint}(F) \cap V$. Finally, by Corollary 5, $x_\alpha + y_0 \in \mathfrak{b}_{FL}(\alpha)$. □

This is the local contour-crossing mechanism used throughout the rest of this work: when a contour crosses the face F, the change is measured exactly by a dimension $\dim F$ vertical contour integral of the residue with a base point on F.

Definition 46. *Let $\mathcal{P}$ be a locally finite real affine hyperplane arrangement. For each positive codimension face F, let V_F denote the localized real open neighborhood from Lemma 6, and set*

$$U_F := V_F + i\mathbb{R}^k.$$

Define $\Theta(\mathbb{C}^k, \mathcal{P}) \subseteq \widehat{R}(\mathbb{C}^k, \mathcal{P})$ to be the vector space of functions f satisfying the following three conditions:

1. *For every positive codimension face F and every flag FL with base F, the following non-vertical contour-vanishing condition holds. The rectangular contours $\Gamma^{\alpha}_{\epsilon,s}(y)$ from Definition 43, formed using the affine residue setting for F, FL, and U_F, satisfy*

$$\lim_{s\to\infty} \int_{\Gamma^{\alpha}_{\epsilon,s}(y)} f\nu = 0$$

   for all $\alpha \in \{\pm 1, \pm i\}^r \setminus \{\pm 1\}^r$, $\epsilon \in \mathbb{R}^r_{>0}$ sufficiently small and separated, $y \in (\mathrm{relint}(F) + i\,\mathrm{lin}(F)) \cap U_F$, and $\nu \in \Lambda^r H^*_{\mathbb{C}}$, where $r = \mathrm{codim}(F)$, $H = \mathrm{lin}(F)^{\perp}$, and $H_{\mathbb{C}} = H + iH$.

2. *$\int_{c+i\mathbb{R}^k} |f\mu| < \infty$ for all $c \in \mathbb{R}^k \setminus \mathcal{P}$ and $\mu \in \Lambda^k(\mathbb{C}^k)^*$.*

3. *The function $c \mapsto \int_{c+i\mathbb{R}^k} f\mu$ is locally constant on $\mathbb{R}^k \setminus \mathcal{P}$ for all $\mu \in \Lambda^k(\mathbb{C}^k)^*$.*

If $\mathcal{P}$ has no positive codimension faces, condition (1) is void. Otherwise, equivalently, $\Theta(\mathbb{C}^k, \mathcal{P}) \subseteq \widehat{R}(\mathbb{C}^k, \mathcal{P})$ is the vector space of functions f such that

$$f|_{U_F} \in \Theta_{FL}(U_F, \mathcal{P})$$

for every positive codimension face F and every flag FL with base F, where $\Theta_{FL}(U_F, \mathcal{P})$ is the local space from Definition 44.

For this equivalence, one direction is just the unpacking of Definition 44: the three conditions above restrict to the three defining conditions of $\Theta_{FL}(U_F, \mathcal{P})$. Conversely, the restriction formulation gives the non-vertical contour-vanishing condition directly, and it gives integrability and local constancy chamber by chamber because Lemma 7 places each chamber inside some localized open neighborhood V_F.

The space $\Theta(\mathbb{C}^k, \mathcal{P})$ packages the local hypotheses needed for every face at once, so that we may later apply the jump formula at every face.

Corollary 6 (Global jump formula)**.** *(→ proof) Let $\mathcal{P}$ be a locally finite real affine hyperplane arrangement, and let $f \in \Theta(\mathbb{C}^k, \mathcal{P})$. Let F be a face of positive codimension, and let*

$$FL = (F_r, \ldots, F_0)$$

be a flag of faces with base $F = F_r$ and with $F_0 = \overline{\mathfrak{c}}$ for some chamber $\mathfrak{c}$. Let $\mu \in \Lambda^k(\mathbb{C}^k)^*$. Then for any $c(\alpha) \in \mathfrak{b}_{FL}(\alpha)$ and any $c \in \mathrm{relint}(F)$,

$$\sum_{\alpha\in\{\pm 1\}^r} \frac{\mathrm{sign}(\alpha)}{(2\pi i)^r} \int_{c(\alpha)+i\mathbb{R}^k} f\mu = \int_{c+i\,\mathrm{lin}(F)} \mathrm{Res}_{FL}(f\mu).$$

Here V_F denotes the localized real open neighborhood from Lemma 6, and the residue is computed in the affine residue setting for $U_F := V_F + i\mathbb{R}^k$.

Proof of Corollary 6. Let V_F be the localized real open neighborhood from Lemma 6, and set $U_F := V_F + i\mathbb{R}^k$. By the equivalent restriction formulation in Definition 46, $f|_{U_F} \in \Theta_{FL}(U_F, \mathcal{P})$, and Lemma 6 gives the hypotheses of Definition 39 for V_F and U_F.

Corollary 5 and the openness of V_F show that $\mathfrak{b}_{FL}(\alpha) \cap V_F \neq \emptyset$ for every α. Choose $c_F(\alpha) \in \mathfrak{b}_{FL}(\alpha) \cap V_F$. Applying Theorem 11 to these points gives

$$\sum_{\alpha \in \{\pm 1\}^r} \frac{\operatorname{sign}(\alpha)}{(2\pi i)^r} \int_{c_F(\alpha)+i\mathbb{R}^k} f\mu = \int_{c+i\,\mathrm{lin}(F)} \mathrm{Res}_{FL}(f\mu).$$

For each α, the points $c_F(\alpha)$ and $c(\alpha)$ lie in the same chamber $\mathfrak{b}_{FL}(\alpha)$. By the local constancy condition in Definition 46, the corresponding integrals are equal. This gives the desired formula. □

6.3 Accessible sufficient conditions for Θ

The definition of $\Theta(\mathbb{C}^k, \mathcal{P})$ is tailored to the precise hypothesis needed for the proof of the jump formula, but it is not always convenient to verify directly in applications. This section makes the jump formula more accessible by giving a simple sufficient condition for a function in $\widehat{R}(\mathbb{C}^k, \mathcal{P})$ to lie in $\Theta(\mathbb{C}^k, \mathcal{P})$, in Proposition 14.

This section is structured as a series of technical lemmas, which we combine to give Proposition 14.

For subsets A, B of $\mathbb{R}^k$ or $\mathbb{C}^k$, we write

$$\mathrm{dist}(A, B) := \inf\{|a - b| : a \in A,\ b \in B\},$$

where $|\cdot|$ denotes the Euclidean norm. By convention, if A or B is empty, $\mathrm{dist}(A, B) = \infty$. Also, if $A = \{a\}$ is a singleton, we write $\mathrm{dist}(a, B)$ instead of $\mathrm{dist}(\{a\}, B)$.

Lemma 28. *(→ proof) Let $V \subseteq \mathbb{R}^k$ be open and connected. Let f be holomorphic on $V + i\mathbb{R}^k$. Suppose that f is bounded on $V + i\mathbb{R}^k$, that for every compact $K \subseteq V$,*

$$\lim_{|y|\to\infty} \sup_{c \in K} |f(c + iy)| = 0,$$

and that f is integrable over $c + i\mathbb{R}^k$ for any $c \in V$.

Then $c \to \int_{c+i\mathbb{R}^k} f(z)\Pi_{\mathbb{C}^k}$ is constant on V.

Proof of Lemma 28. It is enough to prove that the integral is unchanged when the real base point varies in one coordinate direction inside V. Indeed, any two points in a connected open subset of $\mathbb{R}^k$ can be joined by a polygonal path made of such coordinate segments. Fix such a segment and fix the remaining real coordinates. By Fubini's theorem, for almost every value of the remaining imaginary coordinates, the one-variable vertical slices at the two endpoints are integrable. Applying the one-variable argument below to these slices and then integrating over the remaining imaginary variables proves the claim. Thus it is sufficient to assume $k = 1$, so $V \subseteq \mathbb{R}$.

Let $a, b \in V$. Without loss of generality, assume $a < b$. Since V is connected, $[a, b] \subsetneq V$. In this one-dimensional reduction, $\Pi_{\mathbb{C}}$ denotes the standard holomorphic volume form on $\mathbb{C}$.

Let $c = \frac{a+b}{2}$ and $\epsilon = \frac{b-a}{2}$. Let $B_\epsilon(c)$ be the complex ball with radius ϵ centered at c, with boundary oriented counter-clockwise. For each $s > 0$, let $\Gamma_{\epsilon,s}(c)$ be the rectangle with corners $c \pm \epsilon \pm is\epsilon$, with boundary oriented counter-clockwise. We have both $\overline{B_\epsilon(c)} \subsetneq V + i(-\infty, \infty)$ and $\overline{\Gamma_{\epsilon,s}(c)} \subsetneq V + i(-\infty, \infty)$. Further, $\partial\Gamma_{\epsilon,s}(c)$ and $\partial B_\epsilon(c)$ are homotopic in $V + i(-\infty, \infty)$.

For each $\alpha \in \{\pm 1, \pm i\}$, let

$$\Gamma^\alpha_{\epsilon,s}(c) = \begin{cases} \{c + \alpha\epsilon + it : t \in [-s\epsilon, s\epsilon]\} & : \alpha = \pm 1 \\ \{c + t + \alpha s\epsilon : t \in [-\epsilon, \epsilon]\} & : \alpha = \pm i \end{cases}$$

oriented so that $\sum_{\alpha\in\{\pm1,\pm i\}} \Gamma^\alpha_{\epsilon,s}(c) = \Gamma_{\epsilon,s}(c)$ as 1-chains. Then

$$\begin{aligned} 0 &= \int_{\partial B_\epsilon(c)} f\,\Pi_{\mathbb{C}} && \text{Cauchy's theorem} \\ &= \int_{\Gamma_{\epsilon,s}(c)} f\,\Pi_{\mathbb{C}} && \text{Homotopic} \\ &= \sum_{\alpha\in\{\pm1,\pm i\}} \int_{\Gamma^\alpha_{\epsilon,s}(c)} f\,\Pi_{\mathbb{C}} && \text{Equal as 1-chains.} \end{aligned}$$

Take the limit as $s \to \infty$. Suppose $\alpha = \pm i$. The real parts of points on $\Gamma^\alpha_{\epsilon,s}(c)$ lie in the compact interval $[a, b]$, while their imaginary parts have absolute value $s\epsilon$. By the compact-uniform decay hypothesis,

$$\lim_{s\to\infty} \int_{\Gamma^\alpha_{\epsilon,s}(c)} f\,\Pi_{\mathbb{C}} = 0.$$

If $\alpha = \pm 1$, because f is integrable over $\{c + \alpha\epsilon + i\mathbb{R}\}$, dominated convergence gives

$$\lim_{s\to\infty} \int_{\Gamma^\alpha_{\epsilon,s}(c)} f\,\Pi_{\mathbb{C}} = \alpha \int_{c+\alpha\epsilon+i\mathbb{R}} f\,\Pi_{\mathbb{C}}.$$

Note that the sign α accounts for the orientation. Then

$$0 = \sum_{\alpha\in\{\pm1\}} \alpha \int_{c+\alpha\epsilon+i\mathbb{R}} f\,\Pi_{\mathbb{C}},$$

or in other words,

$$\int_{a+i\mathbb{R}} f\,\Pi_{\mathbb{C}} = \int_{b+i\mathbb{R}} f\,\Pi_{\mathbb{C}}.$$

□

Lemma 29. *(→ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement. Then for all $z \in \mathbb{C}^k$,*

$$\operatorname{dist}(z, \mathcal{P}_{\mathbb{C}}) \geq \operatorname{dist}(Re(z), \mathcal{P}).$$

Proof of Lemma 29. If $\mathcal{P} = \emptyset$, then $\mathcal{P}_{\mathbb{C}} = \emptyset$, so by convention

$$\operatorname{dist}(z, \mathcal{P}_{\mathbb{C}}) = \operatorname{dist}(\operatorname{Re}(z), \mathcal{P}) = \infty.$$

Thus the claim is immediate. Otherwise, let $y \in \mathcal{P}_{\mathbb{C}}$ and $z \in \mathbb{C}^k$. Because $\mathcal{P}_{\mathbb{C}}$ is the complexification of a locally finite real affine hyperplane arrangement, $\operatorname{Re}(y) \in \mathcal{P}$. Then $|z - y| \geq |\operatorname{Re}(z) - \operatorname{Re}(y)| \geq \inf_{a\in\mathcal{P}} |\operatorname{Re}(z) - a| = \operatorname{dist}(\operatorname{Re}(z), \mathcal{P})$. Then $\operatorname{dist}(z, \mathcal{P}_{\mathbb{C}}) = \inf_{y\in\mathcal{P}_{\mathbb{C}}} |z - y| \geq \operatorname{dist}(\operatorname{Re}(z), \mathcal{P})$. □

Lemma 30. *(→ proof) Assume the setting of Definition 39. Let $y \in F_{\mathbb{C}}$ with $Re(y) \in \operatorname{relint}(F)$. Let $\tau > \tau_0$. For all $\epsilon \in \mathbb{R}^r_{>0}$ that is sufficiently small and τ-separated and all $\alpha \in \{\pm 1, \pm i\}^r$,*

$$\operatorname{dist}(\Gamma^{\alpha}_{\epsilon,\infty}(y), \mathcal{P}_{\mathbb{C}}) > 0$$

Proof of Lemma 30. Let $z \in \Gamma^{\alpha}_{\epsilon,\infty}(y)$. Then $z = u + y$ for some $u \in H_{\mathbb{C}}$, and $z \in \Gamma^{\alpha}_{\epsilon,s}(y)$ for some $s \geq 1$. Let $q \in \mathfrak{Q}(F, \mathcal{P})$. If $\alpha_{\operatorname{gi}(q)} = \pm 1$, then u satisfies the real-part case of Lemma 27 with the scale vector ϵ. If $\alpha_{\operatorname{gi}(q)} = \pm i$, then u satisfies the imaginary-part case with the scale vector $s\epsilon$, which is still τ-separated. In both cases,

$$|q(z)| \geq |\operatorname{gc}(q)| \epsilon_{\operatorname{gi}(q)} (1 - \frac{\tau_0}{\tau}).$$

Choose a finite set $\mathcal{Q}_F \subseteq \mathfrak{Q}(F, \mathcal{P})$ containing one representative for each hyperplane in $\{Z(q) : q \in \mathfrak{Q}(F, \mathcal{P})\}$. If L_q is the Lipschitz constant of q, then

$$\operatorname{dist}(z, Z_{\mathbb{C}}(q)) \geq \frac{|q(z)|}{L_q}$$

for $q \in \mathcal{Q}_F$. Since $\mathcal{Q}_F$ is finite and $1 - \tau_0/\tau > 0$, we obtain a constant $c_0 > 0$ such that

$$\operatorname{dist}(z, \bigcup_{q \in \mathfrak{Q}(F,\mathcal{P})} Z_{\mathbb{C}}(q)) \geq c_0$$

for all $z \in \Gamma^{\alpha}_{\epsilon,\infty}(y)$. Thus

$$\operatorname{dist}(\Gamma^{\alpha}_{\epsilon,\infty}(y), \bigcup_{q \in \mathfrak{Q}(F,\mathcal{P})} Z_{\mathbb{C}}(q)) > 0.$$

Now consider $\mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})$. Since $\operatorname{Re}(y) \in \operatorname{relint}(F)$ and $\mathcal{P}$ is locally finite, there is a real open neighborhood of $\operatorname{Re}(y)$ meeting no hyperplanes except those that contain F. For ϵ sufficiently small, the compact set $\operatorname{Re}(\Gamma^{\alpha}_{\epsilon,\infty}(y))$ lies in this open neighborhood. Hence

$$\bigcup_{q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F,\mathcal{P})} Z(q) \cap \operatorname{Re}(\Gamma^{\alpha}_{\epsilon,\infty}(y)) = \emptyset.$$

Further, $\operatorname{Re}(\Gamma^{\alpha}_{\epsilon,\infty}(y))$ is compact and $\bigcup_{q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F,\mathcal{P})} Z(q)$ is closed. So

$$\operatorname{dist}(\operatorname{Re}(\Gamma^{\alpha}_{\epsilon,\infty}(y)), \bigcup_{q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F,\mathcal{P})} Z(q)) > 0.$$

Then by Lemma 29

$$\operatorname{dist}(\Gamma^{\alpha}_{\epsilon,\infty}(y), \bigcup_{q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F,\mathcal{P})} Z_{\mathbb{C}}(q)) \geq \operatorname{dist}(\operatorname{Re}(\Gamma^{\alpha}_{\epsilon,\infty}(y)), \bigcup_{q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F,\mathcal{P})} Z(q)) > 0.$$

Because $\mathcal{P}_{\mathbb{C}} = \bigcup_{q \in \mathfrak{Q}(F,\mathcal{P})} Z_{\mathbb{C}}(q) \cup \bigcup_{q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F,\mathcal{P})} Z_{\mathbb{C}}(q)$,

$$\begin{aligned}\operatorname{dist}(\Gamma^{\alpha}_{\epsilon,\infty}(y), \mathcal{P}_{\mathbb{C}}) &\geq \min\Big(\operatorname{dist}(\Gamma^{\alpha}_{\epsilon,\infty}(y), \bigcup_{q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F,\mathcal{P})} Z_{\mathbb{C}}(q)), \\ &\qquad \operatorname{dist}(\Gamma^{\alpha}_{\epsilon,\infty}(y), \bigcup_{q \in \mathfrak{Q}(F,\mathcal{P})} Z_{\mathbb{C}}(q))\Big) \\ &> 0\end{aligned}$$

□

Lemma 31. *(→ proof) Assume the notation of Definition 43. Let $\alpha \in \{\pm 1, \pm i\}^r \setminus \{\pm 1\}^r$. There exists $c_0 > 0$ such that for all $z \in \Gamma^\alpha_{\epsilon,s}(y)$ and $s \geq 0$,*

$$|Im(z)| \geq c_0 s.$$

Proof of Lemma 31. Let $B := \{i \in [1,r] : \alpha_i = \pm i\}$. Since $\alpha \notin \{\pm 1\}^r$, we have $B \neq \emptyset$.

Write $z = \Phi^{-1}(w) + y$, so $\mathrm{Im}(z) = \mathrm{Im}(\Phi^{-1}(w)) + \mathrm{Im}(y)$ with $\mathrm{Im}(\Phi^{-1}(w)) \in H$ and $\mathrm{Im}(y) \in \mathrm{lin}(F)$. Let $\pi_H : \mathbb{R}^k = H \oplus \mathrm{lin}(F) \to H$ be the projection. Since π_H is continuous, there is $C_H > 0$ such that $|\pi_H(v)| \leq C_H |v|$ for all $v \in \mathbb{R}^k$. Thus $|\mathrm{Im}(z)| \geq C_H^{-1} |\mathrm{Im}(\Phi^{-1}(w))|$. Pick any $j \in B$. Then $|u_j^*(\mathrm{Im}(\Phi^{-1}(w)))| = |\mathrm{Im}(w_j)| = s\epsilon_j$, so

$$|\mathrm{Im}(z)| \;\geq\; \frac{1}{C_H} \frac{|u_j^*(\mathrm{Im}(\Phi^{-1}(w)))|}{\|u_j^*\|_{\mathrm{op}}} \;=\; \frac{s\epsilon_j}{C_H \|u_j^*\|_{\mathrm{op}}}.$$

Set $c_0 := \frac{\epsilon_j}{C_H \|u_j^*\|_{\mathrm{op}}}$. □

In the following volume estimate, $\mathcal{H}^r$ denotes r-dimensional Hausdorff measure.

Lemma 32. *(→ proof) Assume the notation of Definition 43. Let $\alpha \in \{\pm 1, \pm i\}^r$ and set $A = \{i : \alpha_i = \pm 1\}$. There exists $c_1 > 0$ such that*

$$\int_{\Gamma^\alpha_{\epsilon,s}(y)} d\mathcal{H}^r = c_1 s^{|A|}$$

for all $s > 0$.

Proof of Lemma 32. Let $B := [1,r] \setminus A$ and let $w = \Phi(z - y)$. $\Gamma^\alpha_{\epsilon,s}(y)$ is a product of segments in coordinates $w_1, \ldots, w_r$. For $i \in A$, w_i ranges over a segment of length $2s\epsilon_i$. For $i \in B$, w_i ranges over a segment of length $2\epsilon_i$. The Jacobian of Φ^{-1} is a constant.

$$\begin{aligned}
\int_{\Gamma^\alpha_{\epsilon,s}(y)} d\mathcal{H}^r &= \int_{\gamma^{\alpha_1}_{\epsilon_1,s} \times \cdots \times \gamma^{\alpha_r}_{\epsilon_r,s}} |\det D\Phi^{-1}| \, d\mathcal{H}^r(w) \\
&= \Big(\prod_{i \in A} 2s\epsilon_i \Big) \Big(\prod_{i \in B} 2\epsilon_i \Big) |\det D\Phi^{-1}| \\
&=: c_1 s^{|A|}.
\end{aligned}$$

□

Lemma 33. *(→ proof) Assume the setting of Definition 44. Let $g \in \widehat{R}(U, \mathcal{P})$. Suppose that for every $E \subseteq U$ such that $Re(E)$ is compact and $\mathrm{dist}(E, \mathcal{P}_{\mathbb{C}}) > 0$, there is $\delta > 0$ such that*

$$\sup_{z \in E} (1 + |Im(z)|)^{k+\delta} |g(z)| < \infty.$$

Then $g \in \Theta_{FL}(U, \mathcal{P})$.

Proof of Lemma 33. First fix $c \in V \setminus \mathcal{P}$ and set $E = c + i\mathbb{R}^k$. Since $\mathcal{P}$ is locally finite and $c \notin \mathcal{P}$, $\operatorname{dist}(c, \mathcal{P}) > 0$. Lemma 29 gives $\operatorname{dist}(E, \mathcal{P}_{\mathbb{C}}) > 0$. Hence, for some $C > 0$ and $\delta > 0$,

$$|g(c + iy)| \leq C(1 + |y|)^{-(k+\delta)}$$

for all $y \in \mathbb{R}^k$. Thus g is integrable on $c + i\mathbb{R}^k$ against $\Pi_{\mathbb{C}^k}$, and hence against every $\mu \in \Lambda^k(\mathbb{C}^k)^*$.

Next choose an open ball B with $c \in B$ and $\overline{B} \subseteq V \setminus \mathcal{P}$. Then $\operatorname{dist}(\overline{B} + i\mathbb{R}^k, \mathcal{P}_{\mathbb{C}}) > 0$ by Lemma 29. The decay hypothesis on $\overline{B} + i\mathbb{R}^k$ gives the compact-uniform decay required in Lemma 28. Since g is holomorphic on $B + i\mathbb{R}^k$, the map

$$c' \mapsto \int_{c'+i\mathbb{R}^k} g \, \Pi_{\mathbb{C}^k}$$

is constant on B. As $\Lambda^k(\mathbb{C}^k)^*$ is one-dimensional, the same local constancy holds for every $\mu \in \Lambda^k(\mathbb{C}^k)^*$.

It remains to check condition (1) of Definition 44. Use the flag coordinates from Definition 39 to construct the rectangular contours. Let τ_0 be the separation constant for these coordinates, let $\tau > \tau_0$, and fix $\epsilon \in \mathbb{R}^r_{>0}$ sufficiently small and τ-separated, $\alpha \in \{\pm 1, \pm i\}^r \setminus \{\pm 1\}^r$, $y \in (\operatorname{relint}(F) + i \operatorname{lin}(F)) \cap U$, and $\nu \in \Lambda^r H^*_{\mathbb{C}}$. Shrinking ϵ if necessary, we may assume $\Gamma^{\alpha}_{\epsilon,\infty}(y) \subseteq U$. Set $A := \{i : \alpha_i = \pm 1\}$. By Lemma 30,

$$\operatorname{dist}(\Gamma^{\alpha}_{\epsilon,\infty}(y), \mathcal{P}_{\mathbb{C}}) > 0.$$

Since $\operatorname{Re}(\Gamma^{\alpha}_{\epsilon,\infty}(y))$ is compact, the decay hypothesis gives $C_2 > 0$ and $\delta > 0$ such that

$$|g(z)| \leq C_2(1 + |\operatorname{Im}(z)|)^{-(k+\delta)}$$

on $\Gamma^{\alpha}_{\epsilon,\infty}(y)$. By Lemma 31, $|\operatorname{Im}(z)| \geq c_0 s$ on $\Gamma^{\alpha}_{\epsilon,s}(y)$, and by Lemma 32,

$$\int_{\Gamma^{\alpha}_{\epsilon,s}(y)} d\mathcal{H}^r = c_1 s^{|A|}.$$

Because ν is a constant r-form, there is $C_\nu > 0$ such that the total variation measure induced by ν on each rectangular face is bounded by $C_\nu \, d\mathcal{H}^r$. Therefore

$$\Big| \int_{\Gamma^{\alpha}_{\epsilon,s}(y)} g \, \nu \Big| \leq \frac{C_\nu C_2 c_1 s^{|A|}}{(1 + c_0 s)^{k+\delta}}.$$

Since $|A| \leq r - 1 \leq k - 1 < k + \delta$, the right-hand side tends to 0 as $s \to \infty$. □

Proposition 14. *(→ proof) Let $\mathcal{P}$ be a locally finite real affine hyperplane arrangement. Let $f \in \widehat{R}(\mathbb{C}^k, \mathcal{P})$. Suppose that for all $E \subseteq \mathbb{C}^k$ such that $Re(E)$ is compact in the topology of $\mathbb{R}^k$ and $\operatorname{dist}(E, \mathcal{P}_{\mathbb{C}}) > 0$, there is $\delta > 0$ such that*

$$\sup_{z \in E} (1 + |Im(z)|)^{k+\delta} |f(z)| < \infty.$$

Then $f \in \Theta(\mathbb{C}^k, \mathcal{P})$.

Proof of Proposition 14. Let $c \in \mathbb{R}^k \setminus \mathcal{P}$. Set $E = c + i\mathbb{R}^k$. Then by assumption, for some $\delta > 0$,

$$C := \sup_{z \in E} (1 + |\mathrm{Im}(z)|)^{k+\delta} |f(z)| < \infty.$$

Thus

$$\int_{c+i\mathbb{R}^k} |f| \, |\Pi_{\mathbb{C}^k}| \leq C \int_{\mathbb{R}^k} (1 + |y|)^{-(k+\delta)} dy < \infty,$$

so f is integrable on $c + i\mathbb{R}^k$ against $\Pi_{\mathbb{C}^k}$. Since $\Lambda^k(\mathbb{C}^k)^*$ is one-dimensional, this proves the integrability condition in Definition 46.

Choose an open ball B containing c with $\overline{B} \subsetneq \mathbb{R}^k \setminus \mathcal{P}$. Because $\overline{B}$ is compact and $\mathcal{P}$ is closed, $\mathrm{dist}(\overline{B}, \mathcal{P}) > 0$. Then by Lemma 29, $\mathrm{dist}(\overline{B} + i\mathbb{R}^k, \mathcal{P}_{\mathbb{C}}) > 0$. Applying the hypothesis to $E = \overline{B} + i\mathbb{R}^k$ gives constants $C_B > 0$ and $\delta_B > 0$ such that

$$|f(c' + iy)| \leq C_B (1 + |y|)^{-(k+\delta_B)}$$

for all $c' \in \overline{B}$ and $y \in \mathbb{R}^k$. This estimate gives boundedness on $B + i\mathbb{R}^k$, integrability over every $c' + i\mathbb{R}^k$ with $c' \in B$, and compact-uniform decay in the imaginary direction. Thus the hypotheses of Lemma 28 hold on B, so $c \mapsto \int_{c+i\mathbb{R}^k} f \, \Pi_{\mathbb{C}^k}$ is constant on B. Since c was arbitrary and $\Lambda^k(\mathbb{C}^k)^*$ is one-dimensional, $c \mapsto \int_{c+i\mathbb{R}^k} f \, \mu$ is locally constant on $\mathbb{R}^k \setminus \mathcal{P}$ for every $\mu \in \Lambda^k(\mathbb{C}^k)^*$. This proves the local constancy condition in Definition 46.

It remains to check the local face condition in Definition 46. Fix a face F of positive codimension and a flag FL with base F. Let $U_F = V_F + i\mathbb{R}^k$ be the localized open neighborhood from Lemma 6. The decay hypothesis for f restricts directly to every $E \subseteq U_F$ with compact real part and positive distance from $\mathcal{P}_{\mathbb{C}}$. Therefore Lemma 33, applied in the affine local setting of Definition 39, gives

$$f|_{U_F} \in \Theta_{FL}(U_F, \mathcal{P}).$$

Since F and FL were arbitrary, $f \in \Theta(\mathbb{C}^k, \mathcal{P})$. □

Proposition 14 is an easy entry point for the jump formula. We will prove this decay estimate, and then the jump formula becomes available at every positive codimension face.

7 Graph traversal algorithm

The jump formula tells us how to cross a chosen face given a flag, and we have defined a face and a flag for each chamber. This determines the directed graph in Definition 26 which organizes our choice of transitions from a given chamber to a set of neighboring chambers. We prove that this graph is acyclic, except for self-cycles. Further, long enough paths eventually push the remainder contour into regions where $\lambda(\mathfrak{c}, \omega)$ is arbitrarily large. Consequently, by Proposition 15, every remainder contour will have arbitrarily fast exponential decay.

First, to check that the graph is ultimately well defined, we need to check that ω' from Lemma 4 exists.

Proof of Lemma 4. Let F be a face of $\mathcal{P}$ of positive codimension and let $F' \neq F'' \in \Xi(F)$. Then

$$H_{F',F''} := \{\omega \in (\mathbb{R}^k)^* : \omega(F', F) = \omega(F'', F)\}$$

is a proper hyperplane in $(\mathbb{R}^k)^*$. Define

$$E(\mathcal{P}) := \{0\} \cup \bigcup_{F} \bigcup_{F' \neq F'' \in \Xi(F)} H_{F',F''},$$

where F ranges over all faces of positive codimension in $\mathcal{P}$. $\mathbb{R}^k$ is σ-compact, so let $\bigcup_\alpha U_\alpha$ be a countable cover of $\mathbb{R}^k$ by compact sets. Because $\mathcal{P}$ is locally finite, there are finitely many faces in each U_α, so there are countably many faces in $\mathbb{R}^k$. So $E(\mathcal{P})$ is a countable union of proper hyperplanes together with $\{0\}$. Each such set has measure zero, and countable unions of measure zero sets have measure zero, so $E(\mathcal{P})$ has measure zero. If $\omega \notin E(\mathcal{P})$, no two values $\omega(F', F)$ with $F' \neq F'' \in \Xi(F)$ are equal. This gives pairwise distinctness, and hence uniqueness of the maximum over any subset of $\Xi(F)$. □

7.1 Strict increase along edges

In this section, we verify that every edge $(\mathfrak{c}, \mathfrak{b})$ with $\lambda(\mathfrak{c}, \omega) < \infty$ improves the potential in Definition 25, which forces acyclicity, except for self-cycles.

Recall the setting of Proposition 4. We have a chamber $\mathfrak{c} \in \Omega(\mathcal{P})$ and a face $F \subseteq \bar{\mathfrak{c}}$ in $\mathcal{P}$. We also have a flag

$$F = F_r \subsetneq \ldots \subsetneq F_0 = \bar{\mathfrak{c}}.$$

Denote this flag by FL. Then for each $\alpha \in \{\pm 1\}^r$, Proposition 4 gives us another flag

$$F(\alpha) = F_r(\alpha) \subsetneq \ldots \subsetneq F_0(\alpha) = \overline{\mathfrak{b}_{FL}(\alpha)}.$$

The next two lemmas compare the faces F_i and $F_i(\alpha)$ for $i \in [0, r]$ such that $\alpha_j = 1$ for all $j \geq i+1$. The first lemma shows that the two faces are equal. The second lemma shows that pattern breaks at the largest index i^* where $\alpha_{i^*} = -1$, where F_{i^*-1} and $F_{i^*-1}(\alpha)$ have opposite generators over F_{i^*} and $F_{i^*}(\alpha)$, respectively.

Lemma 34. *(→ proof) Assume the setting of Proposition 4. Suppose $\alpha \neq 1^r$ and let i^* be the largest index such that $\alpha_{i^*} = -1$. Then for all $i \in [i^*, r]$, $F_i = F_i(\alpha)$. In particular, $F_i \subseteq \overline{\mathfrak{b}_{FL}(\alpha)} \cap \bar{\mathfrak{c}}$.*

Proof of Lemma 34. Let $x \in \operatorname{relint}(F)$ and fix $i \in [i^*, r]$. Choose $\epsilon \in \mathbb{R}^r_{>0}$ sufficiently small and τ_0 separated so that Corollary 5 applies to both α and 1^r. Then

$$y_\alpha := x + \sum_{j=i+1}^{r} \epsilon_j \alpha_j u_j \in \operatorname{relint}(F_i(\alpha))$$

and

$$y := x + \sum_{j=i+1}^{r} \epsilon_j u_j \in \operatorname{relint}(F_i(1^r)) = \operatorname{relint}(F_i).$$

Because $i \geq i^*$, every $j \geq i+1$ satisfies $j > i^*$. By the maximality of i^*, $\alpha_j = 1$ for all such j, so $y_\alpha = y$. Therefore $\operatorname{relint}(F_i(\alpha)) \cap \operatorname{relint}(F_i) \neq \emptyset$. Relative interiors of distinct faces are disjoint, hence $F_i = F_i(\alpha)$.

Since $i \in [i^*, r]$ was arbitrary, this proves $F_i = F_i(\alpha)$ for all $i \in [i^*, r]$. The inclusion follows because $F_i(\alpha) \subseteq F_0(\alpha) = \overline{\mathfrak{b}_{FL}(\alpha)}$ and $F_i \subseteq F_0 = \bar{\mathfrak{c}}$. □

Lemma 35. *(→ proof) Assume the setting of Proposition 4. Suppose $\alpha \neq 1^r$ and let i^* be the largest index such that $\alpha_{i^*} = -1$. Let u be the generator of F_{i^*-1} over F_{i^*}. Then $-u$ is the generator of $F_{i^*-1}(\alpha)$ over $F_{i^*}(\alpha)$.*

Proof of Lemma 35. By Lemma 34, $F_i = F_i(\alpha)$ for all $i \in [i^*, r]$. Let $x \in \operatorname{relint}(F)$. By Corollary 5, for all $\epsilon \in \mathbb{R}^r_{>0}$ sufficiently small and τ_0 separated, for all $i \in [0, r]$, $\operatorname{relint}(F_i(\alpha))$ contains the point $x + \sum_{j=i+1}^r \epsilon_j \alpha_j u_j$. In particular,

1. $\operatorname{relint}(F_{i^*}) = \operatorname{relint}(F_{i^*}(\alpha))$ contains the point $y = x + \sum_{j=i^*+1}^r \epsilon_j u_j$.
2. $\operatorname{relint}(F_{i^*-1})$ contains the point $y + \epsilon_{i^*} u$.
3. $\operatorname{relint}(F_{i^*-1}(\alpha))$ contains the point $y - \epsilon_{i^*} u$.

Then by condition (6) in Proposition 3, $-u$ is the generator of $F_{i^*-1}(\alpha)$ over $F_{i^*}(\alpha)$. □

At this point three different flags become relevant, and it is worth separating them explicitly. We start in a chamber $\mathfrak{c} \in \Omega(\mathcal{P})$ such that $\lambda(\mathfrak{c}, \omega) < \infty$ and we start with its associated flag $FL := FL(\mathfrak{c}, \omega)$ as in Definition 18:

$$F(\mathfrak{c}, \omega) = F_r(\mathfrak{c}, \omega) \subsetneq \cdots \subsetneq F_0(\mathfrak{c}, \omega) = \bar{\mathfrak{c}}.$$

Fix $\alpha \in \{\pm 1\}^r \setminus \{1^r\}$. Next, Proposition 4 gives us a new flag:

$$F(\mathfrak{c}, \omega)(\alpha) = F_r(\mathfrak{c}, \omega)(\alpha) \subsetneq \cdots \subsetneq F_0(\mathfrak{c}, \omega)(\alpha) =: \overline{\mathfrak{b}_{FL}(\alpha)}.$$

For notational convenience, we set $\mathfrak{b} := \mathfrak{b}_{FL}(\alpha)$. Finally, $\mathfrak{b}$ has its own associated flag $FL(\mathfrak{b}, \omega)$ as in Definition 18:

$$F(\mathfrak{b}, \omega) = F_r(\mathfrak{b}, \omega) \subsetneq \cdots \subsetneq F_0(\mathfrak{b}, \omega) = \bar{\mathfrak{b}}.$$

Let i^* be the largest index such that $\alpha_{i^*} = -1$. Applying Lemma 34 to the flag $F(\mathfrak{c}, \omega)(\alpha)$, we have $F_i(\mathfrak{c}, \omega) = F_i(\mathfrak{c}, \omega)(\alpha)$ for all $i \in [i^*, r]$. This trivially implies that for all $i \in [i^* + 1, r]$, the generator of $F_{i-1}(\mathfrak{c}, \omega)$ over $F_i(\mathfrak{c}, \omega)$ is the same as the generator of $F_{i-1}(\mathfrak{c}, \omega)(\alpha)$ over $F_i(\mathfrak{c}, \omega)(\alpha)$.

But by Lemma 35, the pattern breaks at i^*, where the generator of $F_{i^*-1}(\mathfrak{c}, \omega)$ over $F_{i^*}(\mathfrak{c}, \omega)$ is the opposite of the generator of $F_{i^*-1}(\mathfrak{c}, \omega)(\alpha)$ over $F_{i^*}(\mathfrak{c}, \omega)(\alpha)$.

Those results compare the first two flags. We want to show an increase in the potential of Definition 25 from $\mathfrak{c}$ to $\mathfrak{b}$, which is to compare the first flag to the third flag. In the case $\lambda(\mathfrak{b}, \omega) > \lambda(\mathfrak{c}, \omega)$, we are already done. Then in the case $\lambda(\mathfrak{b}, \omega) = \lambda(\mathfrak{c}, \omega)$ but $\dim F(\mathfrak{b}, \omega) \neq \dim F(\mathfrak{c}, \omega)$, we must have $\dim F(\mathfrak{b}, \omega) > \dim F(\mathfrak{c}, \omega)$ because $F(\mathfrak{c}, \omega) \subsetneq \overline{\mathfrak{b}}$ by Lemma 34, so we also have an increase in the potential in that case.

Hence in the following two statements, we assume $\lambda(\mathfrak{b}, \omega) = \lambda(\mathfrak{c}, \omega)$ and $\dim F(\mathfrak{b}, \omega) = \dim F(\mathfrak{c}, \omega)$. Lemma 36 is preparation for Corollary 7, which states that either we have an increase in the potential up to index $i^* + 1$, in which case we are done, or we have $F_i(\mathfrak{b}, \omega) = F_i(\mathfrak{c}, \omega)$ for all $i \in [i^* + 1, r]$.

This gives us a comparison between the first and third flags. In the case $F_i(\mathfrak{b}, \omega) = F_i(\mathfrak{c}, \omega)$ for all $i \in [i^* + 1, r]$, this does not yet give us an increase in the potential, but it sets us up to obtain an increase in the potential at index i^*, in the proof of Proposition 5.

For $j \in [1, r]$ and $\mathfrak{d} \in \{\mathfrak{b}, \mathfrak{c}\}$, define the auxiliary objective pair

$$\mathrm{Aux}_j(\mathfrak{d}) := [\mathfrak{P}(F_j(\mathfrak{d}, \omega), \mathfrak{d}, \omega), \mathfrak{P}(F_j(\mathfrak{d}, \omega), \mathfrak{d}, \omega')],$$

and define

$$\mathrm{Prefix}_j(\mathfrak{d}) := [\mathrm{Aux}_r(\mathfrak{d}), \ldots, \mathrm{Aux}_j(\mathfrak{d})].$$

Set $\mathrm{Prefix}_{r+1}(\mathfrak{d}) := []$.

Lemma 36. *(→ proof) Assume the setting of Proposition 4, with the flag $FL := FL(\mathfrak{c}, \omega)$ as in Definition 18. Suppose $\alpha \neq 1^r$ and let i^* be the largest index such that $\alpha_{i^*} = -1$. Let $\mathfrak{b} = \mathfrak{b}_{FL}(\alpha)$. Suppose $\lambda(\mathfrak{b}, \omega) = \lambda(\mathfrak{c}, \omega)$ and $\dim(F(\mathfrak{b}, \omega)) = \dim(F(\mathfrak{c}, \omega))$.*

For all $i \in [i^*, r]$, either

1. *There exists $j \in [i + 1, r]$ such that*

$$\mathrm{Aux}_j(\mathfrak{b}) > \mathrm{Aux}_j(\mathfrak{c})$$

2. *For all $j \in [i, r]$, $F_j(\mathfrak{c}, \omega) = F_j(\mathfrak{b}, \omega)$.*

Proof of Lemma 36. Base case ($i = r$). $\lambda(\mathfrak{b}, \omega) = \lambda(\mathfrak{c}, \omega)$ and $F(\mathfrak{c}, \omega) \subsetneq \overline{\mathfrak{b}}$, so $F(\mathfrak{c}, \omega) \subseteq F(\mathfrak{b}, \omega)$. Then because $\dim(F(\mathfrak{b}, \omega)) = \dim(F(\mathfrak{c}, \omega))$, $F(\mathfrak{c}, \omega) = F(\mathfrak{b}, \omega)$. So $F_r(\mathfrak{c}, \omega) = F(\mathfrak{c}, \omega) = F(\mathfrak{b}, \omega) = F_r(\mathfrak{b}, \omega)$.

Inductive step. Suppose that for some $i \in [i^* + 1, r]$, either there exists $j \in [i + 1, r]$ such that $\mathrm{Aux}_j(\mathfrak{b}) > \mathrm{Aux}_j(\mathfrak{c})$, or $F_j(\mathfrak{c}, \omega) = F_j(\mathfrak{b}, \omega)$ for all $j \in [i, r]$. In the first case, the same $j \in [i + 1, r] \subseteq [(i - 1) + 1, r]$. So the inductive hypothesis is satisfied for $i - 1$.

Otherwise, $F_j(\mathfrak{c}, \omega) = F_j(\mathfrak{b}, \omega)$ for all $j \in [i, r]$. Now we compare

$$\mathrm{Aux}_i(\mathfrak{c}) \text{ and } \mathrm{Aux}_i(\mathfrak{b}).$$

By Lemma 34, because $i-1 \geq i^*$, $F_{i-1}(\mathfrak{c}, \omega) \subseteq \overline{\mathfrak{b}}$. And $F_i(\mathfrak{b}, \omega) = F_i(\mathfrak{c}, \omega) \subsetneq F_{i-1}(\mathfrak{c}, \omega)$. So $F_{i-1}(\mathfrak{c}, \omega)$ over $F_i(\mathfrak{c}, \omega) = F_i(\mathfrak{b}, \omega)$ is feasible for

$$\mathrm{Aux}_i(\mathfrak{b}).$$

At the same time, $F_{i-1}(\mathfrak{c}, \omega)$ over $F_i(\mathfrak{c}, \omega)$ is optimal for $\mathrm{Aux}_i(\mathfrak{c})$. So $\mathrm{Aux}_i(\mathfrak{b}) \geq \mathrm{Aux}_i(\mathfrak{c})$.

Now consider two cases. If

$$\mathrm{Aux}_i(\mathfrak{b}) > \mathrm{Aux}_i(\mathfrak{c}),$$

$i \in [(i-1)+1, r]$, so in this case the inductive hypothesis is satisfied for $i-1$. Otherwise,

$$\mathrm{Aux}_i(\mathfrak{b}) = \mathrm{Aux}_i(\mathfrak{c}).$$

So $F_{i-1}(\mathfrak{c}, \omega)$ is an optimal face for $\mathrm{Aux}_i(\mathfrak{b})$. Since there is a unique optimal face, $F_{i-1}(\mathfrak{b}, \omega) = F_{i-1}(\mathfrak{c}, \omega)$. So the inductive hypothesis is satisfied for $i-1$ in this case as well. □

Corollary 7. *(→ proof) Assume the setting of Lemma 36. One of the following holds:*

1. $$\mathrm{Prefix}_{i^*+1}(\mathfrak{b}) > \mathrm{Prefix}_{i^*+1}(\mathfrak{c})$$

2. $F_i(\mathfrak{c}, \omega) = F_i(\mathfrak{b}, \omega)$ *for all* $i \in [i^*, r]$ *and*

$$\mathrm{Prefix}_{i^*+1}(\mathfrak{b}) = \mathrm{Prefix}_{i^*+1}(\mathfrak{c})$$

Proof of Corollary 7. By Lemma 36 for $i \leftarrow i^*$, either

1. There exists $j \in [i^*+1, r]$ such that

$$\mathrm{Aux}_j(\mathfrak{b}) > \mathrm{Aux}_j(\mathfrak{c})$$

2. For all $j \in [i^*, r]$, $F_j(\mathfrak{c}, \omega) = F_j(\mathfrak{b}, \omega)$.

In the second case, the second condition of the statement is satisfied. Consider the first case. So there exists $i' \in [i^*+1, r]$ such that

$$\mathrm{Aux}_{i'}(\mathfrak{b}) > \mathrm{Aux}_{i'}(\mathfrak{c}).$$

Then we can assume i' is maximal. If $i' = r$, the first conclusion holds. Otherwise, $i' < r$. By maximality,

$$\mathrm{Aux}_{i'}(\mathfrak{b}) > \mathrm{Aux}_{i'}(\mathfrak{c}),$$

but for all $j \in [i'+1, r]$,

$$\mathrm{Aux}_j(\mathfrak{b}) \leq \mathrm{Aux}_j(\mathfrak{c}).$$

Then by applying Lemma 36 for i', the first condition is false, so for all $j \in [i', r]$, $F_j(\mathfrak{c}, \omega) = F_j(\mathfrak{b}, \omega)$. Thus for all $j \in [i' + 1, r]$,

$$\begin{aligned}\mathrm{Aux}_j(\mathfrak{b}) &= [\omega(F_{j-1}(\mathfrak{b}, \omega), F_j(\mathfrak{b})), \omega'(F_{j-1}(\mathfrak{b}, \omega), F_j(\mathfrak{b}, \omega))] \\ &= [\omega(F_{j-1}(\mathfrak{c}, \omega), F_j(\mathfrak{c})), \omega'(F_{j-1}(\mathfrak{c}, \omega), F_j(\mathfrak{c}, \omega))] \\ &= \mathrm{Aux}_j(\mathfrak{c})\end{aligned}$$

So for all $j \in [i' + 1, r]$,

$$\mathrm{Aux}_j(\mathfrak{b}) = \mathrm{Aux}_j(\mathfrak{c}).$$

At the same time,

$$\mathrm{Aux}_{i'}(\mathfrak{b}) > \mathrm{Aux}_{i'}(\mathfrak{c}).$$

So

$$\mathrm{Prefix}_{i'}(\mathfrak{b}) > \mathrm{Prefix}_{i'}(\mathfrak{c}).$$

Since $i' \in [i^* + 1, r]$, appending the trailing entries at indices $i' - 1, \ldots, i^* + 1$ preserves the strict lexicographic inequality, so

$$\mathrm{Prefix}_{i^*+1}(\mathfrak{b}) > \mathrm{Prefix}_{i^*+1}(\mathfrak{c}).$$

□

Lemma 39 provides auxiliary objectives that have negative sign, so when we flip the generator at indices where $\alpha_i = -1$, we will get a strict increase. First, we need some basic results in polyhedral geometry.

In the induction step of Lemma 39, the quantity we need to control is the generator from F_{i-1} to F_{i-2}, which is not itself a candidate in the optimization over F_i. Lemma 38 identifies the unique other intermediate face F'_{i-1} between F_i and F_{i-2}. Its generator is a competing candidate against the chosen optimal generator of F_{i-1} over F_i, and the resulting two-dimensional geometry forces the generator from F_{i-1} to F_{i-2} to have negative ω-value.

Meanwhile, Lemma 37 is used in the proof of Lemma 38 to reduce from the polyhedra case to the polytope case.

Lemma 37. *(→ proof) Let $\mathcal{P}$ be a locally finite real affine hyperplane arrangement and let $\mathfrak{c} \in \Omega(\mathcal{P})$. Let B be a polyhedron, and set $P := \overline{\mathfrak{c}} \cap B$. Then*

$$\pi(G) := G \cap B$$

is an order isomorphism from the faces G of $\overline{\mathfrak{c}}$ such that

$$\mathrm{relint}(G) \cap \mathrm{int}(B) \neq \emptyset$$

to the faces H of P such that

$$\mathrm{relint}(H) \cap \mathrm{int}(B) \neq \emptyset.$$

In particular, for any $x \in \overline{\mathfrak{c}} \cap \mathrm{int}(B)$, π restricts to a lattice isomorphism from the faces of $\overline{\mathfrak{c}}$ that contain x to the faces of P that contain x.

Additionally, $\dim(\pi(G)) = \dim(G)$.

Proof of Lemma 37. Let $\mathcal{P}_B$ be the locally finite arrangement obtained from $\mathcal{P}$ by adjoining the hyperplanes that define B. For each

$$q \in \mathfrak{Q}(\mathcal{P}_B) \setminus \mathfrak{Q}(\mathcal{P}),$$

the hyperplane $Z(q)$ is one of the added hyperplanes, so $Z(q) \cap \operatorname{int}(B) = \emptyset$. Hence $\operatorname{sign}(q(x))$ is constant for $x \in \operatorname{int}(B)$; denote this common value by $\operatorname{sign}_B(q)$.

First, let G be a face of $\bar{\mathfrak{c}}$ such that

$$\operatorname{relint}(G) \cap \operatorname{int}(B) \neq \emptyset.$$

Choose

$$z \in \operatorname{relint}(G) \cap \operatorname{int}(B).$$

Since the intersection of a face with a convex subset is a face of that subset, $G \cap B = G \cap P$ is a face of P. Moreover $\operatorname{int}(B) \cap \operatorname{relint}(G)$ is an open neighborhood of z in $\operatorname{aff}(G)$ contained in $G \cap B$. Thus $\operatorname{aff}(G \cap B) = \operatorname{aff}(G)$, and by the characterization of relative interior as interior in the affine hull, $z \in \operatorname{relint}(G \cap B)$. So $\pi(G) = G \cap B$ lies in the claimed target set. Since $z \in \operatorname{relint}(G \cap B) \cap \operatorname{int}(B)$, for every $q \in \mathfrak{Q}(\mathcal{P}_B)$,

$$\operatorname{sign}_{\pi(G)}(q) = \operatorname{sign}(q(z)) = \begin{cases} \operatorname{sign}_G(q) & : q \in \mathfrak{Q}(\mathcal{P}) \\ \operatorname{sign}_B(q) & : q \in \mathfrak{Q}(\mathcal{P}_B) \setminus \mathfrak{Q}(\mathcal{P}). \end{cases}$$

Now let H be a face of P such that

$$\operatorname{relint}(H) \cap \operatorname{int}(B) \neq \emptyset.$$

Choose

$$y \in \operatorname{relint}(H) \cap \operatorname{int}(B),$$

and let $\rho(H)$ be the unique face of $\bar{\mathfrak{c}}$ such that $y \in \operatorname{relint}(\rho(H))$. Then $\rho(H)$ lies in the claimed source set. For every $q \in \mathfrak{Q}(\mathcal{P})$,

$$\operatorname{sign}_{\rho(H)}(q) = \operatorname{sign}(q(y)) = \operatorname{sign}_H(q).$$

Thus this definition is independent of the choice of y.

We showed already that $y \in \operatorname{relint}(\rho(H)) \cap \operatorname{int}(B)$ implies

$$y \in \operatorname{relint}(\rho(H) \cap B) = \operatorname{relint}(\pi(\rho(H))).$$

Since also $y \in \operatorname{relint}(H)$,

$$\pi(\rho(H)) = H.$$

Similarly, we showed that $z \in \operatorname{relint}(\pi(G))$, while $z \in \operatorname{relint}(G)$, so

$$\rho(\pi(G)) = G.$$

Thus π and ρ are inverse bijections between the two stated collections of faces.

Inclusion of faces is equivalent to reverse inclusion of the affine forms which vanish on them. The displayed sign equalities show that π and ρ preserve which affine forms vanish. So π and ρ are order preserving, and hence π is an order isomorphism.

Next we check that π preserves dimension. Let $z \in \operatorname{relint}(G) \cap \operatorname{int}(B)$. Then $G \cap B$ contains a relative open neighborhood of z in $\operatorname{aff}(G)$, so $\dim(G \cap B) \geq \dim(G)$. Clearly $\dim(G \cap B) \leq \dim(G)$, so $\dim(G \cap B) = \dim(G)$.

If $x \in \overline{\mathfrak{c}} \cap \operatorname{int}(B)$, then every face G of $\overline{\mathfrak{c}}$ containing x has relative interior meeting $\operatorname{int}(B)$. Indeed, $G = \overline{\operatorname{relint}(G)}$ and $\operatorname{int}(B)$ is an open neighborhood of x. The same argument applies to every face of P containing x.

Since $x \in B$, a face G of $\overline{\mathfrak{c}}$ contains x if and only if $\pi(G) = G \cap B$ contains x. Also, if a face H of P contains x, then $H = \pi(\rho(H)) = \rho(H) \cap B$, so $\rho(H)$ contains x. Therefore π restricts to an order isomorphism

$$\{G \leq \overline{\mathfrak{c}} : x \in G\} \longrightarrow \{H \leq P : x \in H\}.$$

These two posets are intervals in their respective face lattices, from the minimal face containing x to the top face. Thus they are lattices, and the order isomorphism between them is a lattice isomorphism. □

Lemma 38. *(→ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement and let $\mathfrak{c} \in \Omega(\mathcal{P})$ be a chamber. Let $F_i \subsetneq F_{i-2} \subseteq \overline{\mathfrak{c}}$ be faces such that $\operatorname{codim}(F_i) = i$ and $\operatorname{codim}(F_{i-2}) = i-2$. Then there are exactly two faces $F_{i-1} \neq F'_{i-1}$ with codimension $i-1$ such that $F_i \subsetneq F_{i-1} \subsetneq F_{i-2}$ and $F_i \subsetneq F'_{i-1} \subsetneq F_{i-2}$.*

Proof of Lemma 38. The diamond property [39, Theorem 2.7(iii)] states this result for polytopes, which are bounded polyhedra. We reduce to this case by localizing.

Choose $x \in \operatorname{relint}(F_i)$. Choose a polytope B with $x \in \operatorname{int}(B)$, for instance a cube, and set

$$P := \overline{\mathfrak{c}} \cap B.$$

Since $\mathcal{P}$ is locally finite and P is compact, P is defined by finitely many linear inequalities. Further, B is bounded, so P is bounded, so P is a polytope. By Lemma 37, the map

$$\pi(G) = G \cap B$$

is a lattice isomorphism from the faces of $\overline{\mathfrak{c}}$ that contain x to the faces of P that contain x. Additionally, π preserves dimension, so the interval $[\pi(F_i), \pi(F_{i-2})]$ has length two.

By the diamond property [39, Theorem 2.7(iii)], there are exactly two faces $H_1 \neq H_2$ of P such that, for $a = 1, 2$,

$$\pi(F_i) \subsetneq H_a \subsetneq \pi(F_{i-2}).$$

Since $x \in F_i \cap B \subsetneq H_a$, each H_a contains x. Set $G_a = \pi^{-1}(H_a)$. Then

$$F_i \subsetneq G_a \subsetneq F_{i-2}.$$

Conversely, if G is a face of $\bar{\mathfrak{c}}$ such that

$$F_i \subsetneq G \subsetneq F_{i-2},$$

then

$$\pi(F_i) \subsetneq \pi(G) \subsetneq \pi(F_{i-2}).$$

Thus $\pi(G) = H_a$ for some $a = 1, 2$, and injectivity of π gives $G = G_a$. □

Lemma 39. *(→ proof) Suppose $\lambda(\mathfrak{c}, \omega) < \infty$ and let $r = \operatorname{codim}(F(\mathfrak{c}, \omega))$. Then $\mathfrak{P}(F_i(\mathfrak{c}, \omega), \mathfrak{c}, \omega) < 0$ for all $i \in [1, r]$.*

Proof of Lemma 39. For notational simplicity, we let $F_i := F_i(\mathfrak{c}, \omega)$. We proceed by induction on i.

In the base case, let $i = r$, so $F_i = F(\mathfrak{c}, \omega)$. There exists a unique $u \in \operatorname{lin}(F_r)^\perp$ such that $|u| = 1$ and every $z \in F_{r-1}$ can be uniquely written as $z = x + tu$ for some $x \in \operatorname{aff}(F_r)$ and $t \in \mathbb{R}_{\geq 0}$. By definition, $\mathfrak{P}(F_r, \mathfrak{c}, \omega) = \omega(u)$.

Because $\operatorname{codim}(F_{r-1}) = \operatorname{codim}(F_r) - 1$, $F_{r-1} \setminus F_r \neq \emptyset$. Thus there is $x \in \operatorname{aff}(F_r)$ and $t > 0$ such that $x + tu \in F_{r-1} \setminus F_r \subseteq \bar{\mathfrak{c}} \setminus F(\mathfrak{c}, \omega)$. Then $\omega(x + tu) < \lambda(\mathfrak{c}, \omega)$. Since $x \in \operatorname{aff}(F_r)$, $\omega(x) = \lambda(\mathfrak{c}, \omega)$. Then $\mathfrak{P}(F_r, \mathfrak{c}, \omega) = \omega(u) < 0$.

Now assume for the inductive hypothesis that for some $i \in [2, r]$, $\mathfrak{P}(F_i, \mathfrak{c}, \omega) < 0$. We are to show that $\mathfrak{P}(F_{i-1}, \mathfrak{c}, \omega) < 0$.

We have $F_i \subsetneq F_{i-1} \subsetneq F_{i-2} \subseteq \bar{\mathfrak{c}}$. By Lemma 38, there exists another face $F'_{i-1} \neq F_{i-1}$ of codimension $i - 1$ such that $F_i \subsetneq F'_{i-1} \subsetneq F_{i-2} \subseteq \bar{\mathfrak{c}}$. Let

1. $u \in \operatorname{lin}(F_{i-2}) \cap \operatorname{lin}(F_{i-1})^\perp$ be the generator of F_{i-2} over F_{i-1},
2. $v \in \operatorname{lin}(F_{i-1}) \cap \operatorname{lin}(F_i)^\perp$ be the generator of F_{i-1} over F_i,
3. $w \in \operatorname{lin}(F'_{i-1}) \cap \operatorname{lin}(F_i)^\perp$ be the generator of F'_{i-1} over F_i.

Choose $x \in \operatorname{relint}(F_i)$. By condition (5) in Proposition 3 for w, after shrinking $\delta > 0$ if necessary we can arrange that $x + \delta w \in \operatorname{relint}(F'_{i-1}) \subsetneq F_{i-2}$. Since $F'_{i-1} \neq F_{i-1}$ and relative interiors of faces are disjoint, $x + \delta w \notin F_{i-1}$. Thus condition (2) in Proposition 3 for u gives $y \in \operatorname{aff}(F_{i-1})$ and $t \in \mathbb{R}_{>0}$ such that $y + tu = x + \delta w$. Because $\operatorname{aff}(F_{i-1}) = \operatorname{aff}(F_i) \oplus \mathbb{R}v$, there exist $z \in \operatorname{aff}(F_i)$ and $s \in \mathbb{R}$ such that $y = z + sv$. Then $z + sv + tu = x + \delta w$. Then $z - x = \delta w - sv - tu$. $z, x \in \operatorname{aff}(F_i)$ and $w, v, u \in \operatorname{lin}(F_i)^\perp$. Then $z - x \in \operatorname{lin}(F_i)$ and $\delta w - sv - tu \in \operatorname{lin}(F_i)^\perp$. Thus $\delta w - sv - tu \in \operatorname{lin}(F_i) \cap \operatorname{lin}(F_i)^\perp$, so $\delta w = sv + tu$. So $w = \frac{sv+tu}{\delta}$. Let $a = \frac{s}{\delta}$ and $b = \frac{t}{\delta}$, so that $w = av + bu$ and $b > 0$.

Since $F'_{i-1} \neq F_{i-1}$, w and v are not co-linear, so $b > 0$. Because $u \in \operatorname{lin}(F_{i-1})^\perp$ and $v \in \operatorname{lin}(F_{i-1})$, u and v are orthonormal. Then because w is unit and $w = av + bu$, $a^2 + b^2 = 1$. Because $b > 0$, it follows that $a < 1$. Next, by optimality of v for $\mathfrak{P}(F_i, \mathfrak{c}, \omega)$, $\omega(v) \geq \omega(w)$. Then, $a\omega(v) + b\omega(u) = \omega(w) \leq \omega(v)$. So $\omega(u) \leq \frac{(1-a)}{b}\omega(v)$. $\omega(v) = \mathfrak{P}(F_i, \mathfrak{c}, \omega)$ by Definition 16, so by assumption $\omega(v) < 0$. Then because $a < 1$ and $b > 0$, $\omega(u) \leq \frac{(1-a)}{b}\omega(v) < 0$. $\omega(u) = \mathfrak{P}(F_{i-1}, \mathfrak{c}, \omega)$ by definition, so $\mathfrak{P}(F_{i-1}, \mathfrak{c}, \omega) < 0$. □

Now we are ready to show that every edge $(\mathfrak{c}, \mathfrak{b})$ with $\lambda(\mathfrak{c}, \omega) < \infty$ improves the potential in Definition 25.

Proof of Proposition 5. Fix $\alpha \in \{\pm 1\}^r \setminus \{1\}^r$ and set $\mathfrak{b} := \mathfrak{b}_{FL}(\alpha)$.

$F(\mathfrak{c}, \omega) \subsetneq \overline{\mathfrak{b}}$ by Proposition 4. So $\lambda(\mathfrak{b}, \omega) \geq \lambda(\mathfrak{c}, \omega)$. If $\lambda(\mathfrak{b}, \omega) > \lambda(\mathfrak{c}, \omega)$, the conclusion holds.

Otherwise, suppose $\lambda(\mathfrak{b}, \omega) = \lambda(\mathfrak{c}, \omega)$. We have $F(\mathfrak{c}, \omega) \subsetneq \overline{\mathfrak{b}}$. Then $F(\mathfrak{c}, \omega) \subseteq F(\mathfrak{b}, \omega)$. Then $\dim(F(\mathfrak{b}, \omega)) \geq \dim(F(\mathfrak{c}, \omega))$. If $\dim(F(\mathfrak{b}, \omega)) > \dim(F(\mathfrak{c}, \omega))$, the conclusion holds.

From now on, we suppose $\dim(F(\mathfrak{b}, \omega)) = \dim(F(\mathfrak{c}, \omega))$. Since $\alpha \neq 1^r$, let i^* be the largest index such that $\alpha_{i^*} = -1$.

By Corollary 7, either

1. $$\mathrm{Prefix}_{i^*+1}(\mathfrak{b}) > \mathrm{Prefix}_{i^*+1}(\mathfrak{c})$$

2. $$F_i(\mathfrak{c}, \omega) = F_i(\mathfrak{b}, \omega) \qquad \forall i \in [i^*, r]$$

   and
   $$\mathrm{Prefix}_{i^*+1}(\mathfrak{b}) = \mathrm{Prefix}_{i^*+1}(\mathfrak{c})$$

In the first case, the conclusion holds. Otherwise suppose the second case.

Because $\alpha_i = 1$ for all $i > i^*$, $F_{i^*}(\mathfrak{c}, \omega)(\alpha) = F_{i^*}(\mathfrak{c}, \omega)$. Then because $F_{i^*}(\mathfrak{b}, \omega) = F_{i^*}(\mathfrak{c}, \omega)$, $F_{i^*}(\mathfrak{b}, \omega) = F_{i^*}(\mathfrak{c}, \omega)(\alpha)$. So $F_{i^*}(\mathfrak{b}, \omega) = F_{i^*}(\mathfrak{c}, \omega)(\alpha) \subsetneq F_{i^*-1}(\mathfrak{c}, \omega)(\alpha)$.

$F_{i^*-1}(\mathfrak{c}, \omega)(\alpha) \subseteq \overline{\mathfrak{b}}$ by Proposition 4. So $F_{i^*}(\mathfrak{b}, \omega) \subsetneq F_{i^*-1}(\mathfrak{c}, \omega)(\alpha) \subseteq \overline{\mathfrak{b}}$. So $F_{i^*-1}(\mathfrak{c}, \omega)(\alpha)$ over $F_{i^*}(\mathfrak{b}, \omega)$ is feasible for $\mathrm{Aux}_{i^*}(\mathfrak{b})$.

Let u be the generator of $F_{i^*-1}(\mathfrak{c}, \omega)$ over $F_{i^*}(\mathfrak{c}, \omega)$. By Lemma 35, $-u$ is the generator of $F_{i^*-1}(\mathfrak{c}, \omega)(\alpha)$ over $F_{i^*}(\mathfrak{c}, \omega)(\alpha)$. Since $F_{i^*}(\mathfrak{c}, \omega)(\alpha) = F_{i^*}(\mathfrak{b}, \omega)$, $-u$ is the generator of $F_{i^*-1}(\mathfrak{c}, \omega)(\alpha)$ over $F_{i^*}(\mathfrak{b}, \omega)$. Thus

$$\begin{aligned}
\mathrm{Aux}_{i^*}(\mathfrak{b}) &\geq [\omega(F_{i^*-1}(\mathfrak{c}, \omega)(\alpha), F_{i^*}(\mathfrak{b}, \omega)), \\
&\qquad \omega'(F_{i^*-1}(\mathfrak{c}, \omega)(\alpha), F_{i^*}(\mathfrak{b}, \omega))] \\
&= [\omega(-u), \omega'(-u)] \\
&= [-\omega(u), -\omega'(u)] \\
&= [-\omega(F_{i^*-1}(\mathfrak{c}, \omega), F_{i^*}(\mathfrak{c}, \omega)), \\
&\qquad -\omega'(F_{i^*-1}(\mathfrak{c}, \omega), F_{i^*}(\mathfrak{c}, \omega))] \\
&= [-\mathfrak{P}(F_{i^*}(\mathfrak{c}, \omega), \mathfrak{c}, \omega), \\
&\qquad -\mathfrak{P}(F_{i^*}(\mathfrak{c}, \omega), \mathfrak{c}, \omega')]
\end{aligned}$$

By Lemma 39,

$$\mathfrak{P}(F_{i^*}(\mathfrak{c}, \omega), \mathfrak{c}, \omega) < 0.$$

Hence

$$\mathfrak{P}(F_{i^*}(\mathfrak{b},\omega),\mathfrak{b},\omega) \geq -\mathfrak{P}(F_{i^*}(\mathfrak{c},\omega),\mathfrak{c},\omega) > 0 > \mathfrak{P}(F_{i^*}(\mathfrak{c},\omega),\mathfrak{c},\omega).$$

□

Now that we have a strict increase in some potential function, we have an acyclic graph, except for self-cycles.

Proof of Corollary 1. For any directed graph $G = (V, E)$ with vertices V and edges E, G is acyclic except for self-cycles if there exists a partially ordered set Λ and a function $f : V \to \Lambda$ such that for all edges $(v, v') \in E$ with $v \neq v'$, $f(v') > f(v)$. Let $\Lambda = (\mathbb{R} \cup \{\infty\})^{2(k+1)}$ with lexicographic ordering. Let $f : \Omega(\mathcal{P}) \to \Lambda$ be defined by $f(\mathfrak{c}) := \mathfrak{P}^*(\mathfrak{c}, \omega)$, which is defined for every chamber by Definition 25. If $(\mathfrak{c}, \mathfrak{b})$ is an edge with $\mathfrak{c} \neq \mathfrak{b}$, then $\lambda(\mathfrak{c}, \omega) < \infty$ by Definition 26, so Proposition 5 gives $f(\mathfrak{b}) > f(\mathfrak{c})$. Therefore $G_\omega(\mathcal{P})$ is acyclic except for self-cycles. □

7.2 Eventual large increase

So far we have not used the assumption that $\mathcal{P}$ has finitely many normal vector spaces. The proof of Proposition 6 is the only place where we use this assumption. So some of our results, and in particular Theorem 12, allow $\mathcal{P}$ to have infinitely many normal vector spaces. However, for the main theorem we want the remainders to vanish to arbitrary exponential rates, which requires the conclusion of Proposition 6 to hold.

Lemma 40. *(→ proof) Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement, and let $\mathfrak{c}, \mathfrak{b} \in \Omega(\mathcal{P})$. If $x \in \overline{\mathfrak{c}} \cap \overline{\mathfrak{b}}$ is a vertex of $\overline{\mathfrak{c}}$, then x is a vertex of $\overline{\mathfrak{b}}$.*

Proof of Lemma 40. Let $\mathcal{H}_x$ be the set of hyperplanes of the arrangement $\mathcal{P}$ which contain x. By local finiteness, there is $\epsilon > 0$ such that every hyperplane of the arrangement which meets $B_\epsilon(x)$ lies in $\mathcal{H}_x$.

We first show that $\bigcap_{H \in \mathcal{H}_x} H = \{x\}$. Otherwise, this intersection contains a line $x + \mathbb{R}u$ with $u \neq 0$. For sufficiently small $t > 0$, the points $x + tu$ and $x - tu$ lie in $B_\epsilon(x)$. They lie on every hyperplane in $\mathcal{H}_x$ and on the same side of every remaining hyperplane as x. Since $x \in \overline{\mathfrak{c}}$, this implies $x + tu, x - tu \in \overline{\mathfrak{c}}$, contradicting that x is a vertex of $\overline{\mathfrak{c}}$.

Now suppose $x = (y + z)/2$ for some $y, z \in \overline{\mathfrak{b}}$. Replacing y, z by $x + \delta(y - x), x + \delta(z - x)$ for sufficiently small $\delta \in (0, 1]$, we may assume $y, z \in B_\epsilon(x)$ while keeping $x = (y + z)/2$. For each $H \in \mathcal{H}_x$, $\overline{\mathfrak{b}}$ is contained in one of the two closed halfspaces bounded by H, and $x \in H$. Thus $y, z \in H$. Hence $y, z \in \bigcap_{H \in \mathcal{H}_x} H = \{x\}$, so $y = z = x$. Therefore x is a vertex of $\overline{\mathfrak{b}}$. □

Proof of Proposition 6. If $\lambda(\mathfrak{c}_0, \omega) = \infty$, then the only outgoing edge from $\mathfrak{c}_0$ is the self-cycle, so every path starting at $\mathfrak{c}_0$ stays at $\mathfrak{c}_0$. Thus we may take $L_0 = 1$ in this case.

Assume $\lambda(\mathfrak{c}_0, \omega) < \infty$. Let $\mathfrak{A} := \{\ell \in (\mathbb{R}^k)^* \setminus \{0\} : \exists s \in \mathbb{R} \text{ s.t. } \ell^{-1}(s) \subseteq \mathcal{P}\}$. Because $\mathcal{P}$ has finitely many normal vector spaces, $\mathfrak{A}$ is finite, up to scaling, and every chamber is a polyhedron. Every constraint of the linear program is of the form $\ell(x) \leq s$ for some $\ell \in \mathfrak{A}$ and $s \in \mathbb{R}$. Further, because $\lambda(\mathfrak{c}_0, \omega) < \infty$, $\omega \in \operatorname{span} \mathfrak{A}$. Represent $\mathbb{R}^k = H \oplus \mathfrak{A}^\perp$ for $\mathfrak{A}^\perp := \{x \in \mathbb{R}^k : \ell(x) = 0 \ \forall \ell \in \mathfrak{A}\}$ and some

complement H. Both the objective and constraints only depend on H. For the rest of this proof, we fix a point in $\mathfrak{A}^\perp$ and work only in H. After doing so, we can assume that $\mathfrak{A}^\perp$ is trivial and $\operatorname{span}\mathfrak{A} = (\mathbb{R}^k)^*$. Then one of the most fundamental results of linear programming [28, Corollary 2.11] is that every chamber $\mathfrak{c}$ with $\lambda(\mathfrak{c},\omega) < \infty$ has an optimal vertex. Fix an optimal vertex x_0 of $\overline{\mathfrak{c}}_0$.

Because $\mathfrak{A}$ is finite, up to scaling, there is a finite number of directions between two neighboring vertices. That is, there exists $J \in \mathbb{Z}_{\geq 1}$ and unit vectors $u_1, \dots, u_J \in \mathbb{R}^k$ such that for all neighboring vertices v and v', $v' = v + tu_j$ for some $t \in \mathbb{R}_{\neq 0}$ and $j \in [1, J]$. Then $\frac{\omega(v')-\omega(v)}{|v'-v|} = \frac{t\omega(u_j)}{|tu_j|} = \operatorname{sign}(t)\omega(u_j)$. Let S be the set of positive values among $\{\omega(u_j), -\omega(u_j) : j \in [1, J]\}$.

For every edge $(\mathfrak{c}, \mathfrak{b})$ in $G_\omega(\mathcal{P})$, we have $\lambda(\mathfrak{b},\omega) \geq \lambda(\mathfrak{c},\omega)$: this is immediate for the self-cycle from a chamber with $\lambda = \infty$, and otherwise follows from $F(\mathfrak{c},\omega) \subseteq \overline{\mathfrak{b}}$.

If $\lambda(\mathfrak{c}_0,\omega) > \eta$, then every path starting at $\mathfrak{c}_0$ has terminal chamber $\mathfrak{c}_L$ with $\lambda(\mathfrak{c}_L,\omega) \geq \lambda(\mathfrak{c}_0,\omega) > \eta$, so we may take $L_0 = 1$ in this case.

Assume from now on that $\lambda(\mathfrak{c}_0,\omega) \leq \eta$. We use the convention $\overline{B}(x, 0) = \{x\}$. If $S = \emptyset$, let $B := \overline{B}(x_0, 0)$. If $S \neq \emptyset$, set $s := \min S$ and let

$$B := \overline{B}\Big(x_0, \frac{\eta - \lambda(\mathfrak{c}_0,\omega)}{s}\Big).$$

Because $\mathcal{P}$ is locally finite, only the closures of finitely many chambers intersect B; let N be the number of such chambers and set $L_0 := N + 1$.

Now let $(\mathfrak{c}_0, \dots, \mathfrak{c}_L)$ be a path of length $L \geq L_0$. Suppose, towards contradiction, that $\lambda(\mathfrak{c}_L,\omega) \leq \eta$. Starting from the fixed vertex x_0, choose vertices inductively as follows. Suppose x_i has been chosen as an optimal vertex of $\overline{\mathfrak{c}}_i$. Then $x_i \in F(\mathfrak{c}_i,\omega) \subseteq \overline{\mathfrak{c}}_{i+1}$, so Lemma 40 shows that x_i is a vertex of $\overline{\mathfrak{c}}_{i+1}$. If x_i is already optimal in $\overline{\mathfrak{c}}_{i+1}$, set $x_{i+1} := x_i$, $L_i := 0$, and $v_{i0} := x_i$. Otherwise, repeatedly applying Lemma 41 and using the fact that polyhedra have finitely many vertices gives a path of neighboring vertices

$$x_i = v_{i0}, \dots, v_{iL_i} = x_{i+1}$$

such that x_{i+1} is optimal in $\overline{\mathfrak{c}}_{i+1}$ and $\omega(v_{il}) > \omega(v_{i(l-1)})$ for all $l \in [1, L_i]$.

If $S = \emptyset$, then the second alternative cannot occur, since its first step would have a positive slope. Hence $x_j = x_0$ for every $j \in [0, L]$, so $x_j \in B$ for every $j \in [0, L]$.

If $S \neq \emptyset$, each nonzero step in the path from x_i to x_{i+1} has positive slope bounded below by s. Hence, for each $i \in [0, L-1]$,

$$\omega(x_{i+1}) - \omega(x_i) = \sum_{l=1}^{L_i} \big(\omega(v_{il}) - \omega(v_{i(l-1)})\big) \geq s \sum_{l=1}^{L_i} |v_{il} - v_{i(l-1)}| \geq s|x_{i+1} - x_i|.$$

Since each x_j is optimal in $\overline{\mathfrak{c}}_j$ and λ is non-decreasing along the path, for every $j \in [0, L]$,

$$s \sum_{i=0}^{j-1} |x_{i+1} - x_i| \leq \sum_{i=0}^{j-1} \big(\omega(x_{i+1}) - \omega(x_i)\big) = \omega(x_j) - \omega(x_0) \leq \omega(x_L) - \omega(x_0) \leq \eta - \lambda(\mathfrak{c}_0,\omega).$$

Thus $|x_j - x_0| \leq \frac{\eta - \lambda(\mathfrak{c}_0,\omega)}{s}$, so $x_j \in B$ for every $j \in [0, L]$. In either case, $x_j \in B \cap \overline{\mathfrak{c}}_j$ for every $j \in [0, L]$. Hence every $\mathfrak{c}_j$ belongs to the finite set of chambers whose closures meet B. Since $L \geq N + 1$, some chamber repeats along the path, producing a cycle. This cycle is not a self-cycle because all chambers in the path have finite λ, contradicting Corollary 1. Therefore $\lambda(\mathfrak{c}_L, \omega) > \eta$. $\square$

The proof of Proposition 6 essentially iteratively runs the simplex algorithm from an optimal vertex of one chamber to an optimal vertex of another chamber. One of the key ideas of the simplex algorithm is Lemma 41. However, most linear programming references use this idea implicitly. Because they assume their feasible set is written in standard form, we cannot immediately translate their results. So we provide our own statement and proof.

Lemma 41. *(→ proof) Let $C \subseteq \mathbb{R}^k$ be a nonempty polyhedron, and let $\omega \in (\mathbb{R}^k)^*$ be bounded above on C. If v is a vertex of C which is not optimal for maximizing ω, then there is a neighboring vertex v' of C such that*

$$\omega(v') > \omega(v).$$

Proof of Lemma 41. Let K be the cone of feasible directions of C at v:

$$K := \{u \in \mathbb{R}^k : v + \epsilon u \in C \text{ for all sufficiently small } \epsilon > 0\}.$$

If $C = \{x : a_j(x) \leq b_j,\ j = 1, \ldots, N\}$ for linear functions a_j and real numbers b_j, then

$$K = \{u \in \mathbb{R}^k : a_j(u) \leq 0 \text{ for all } j \text{ with } a_j(v) = b_j\},$$

so K is a polyhedral cone. Since v is a vertex, K is pointed: if $u, -u \in K$, then for sufficiently small $\epsilon > 0$ both $v + \epsilon u$ and $v - \epsilon u$ lie in C, and their midpoint is v, forcing $u = 0$. By [31, Section 8.8], $K = \mathbb{R}_{\geq 0}u_1 + \cdots + \mathbb{R}_{\geq 0}u_n$, where $u_1, \ldots, u_n$ are non-zero and each $\mathbb{R}_{\geq 0}u_i$ is a one-dimensional face of K.

Since v is not optimal, there is $y \in C$ such that $\omega(y) > \omega(v)$. Then $y - v \in K$, since $v + \epsilon(y - v) = (1 - \epsilon)v + \epsilon y \in C$ for $0 \leq \epsilon \leq 1$. Write $y - v = \sum_i t_i u_i$ with $t_i \geq 0$. Since $\omega(y - v) > 0$, some one-dimensional face $R = \mathbb{R}_{\geq 0}u_i$ of K satisfies $\omega(u_i) > 0$. Set u as this u_i and set $R = \mathbb{R}_{\geq 0}u$. Set

$$E_R := C \cap (v + R).$$

We claim that E_R is a one-dimensional face of C. Indeed, if $x_1, x_2 \in C$ and $\frac{x_1 + x_2}{2} \in E_R$, then

$$\frac{(x_1 - v) + (x_2 - v)}{2} \in R.$$

Also $x_1 - v, x_2 - v \in K$, since for $i = 1, 2$, by convexity of C, the segment

$$v + \epsilon(x_i - v) = (1 - \epsilon)v + \epsilon x_i$$

lies in C for $0 \leq \epsilon \leq 1$.

Since R is a face of K, it follows that $x_1 - v, x_2 - v \in R$, hence $x_1, x_2 \in E_R$. Thus E_R is a face of C. It is one-dimensional because it is contained in the ray $v + R$, and it contains $v + \epsilon u$ for all sufficiently small $\epsilon > 0$.

Since E_R is a closed convex subset of the ray $v + R$, it is either an unbounded ray $v + \mathbb{R}_{\geq 0}u$ of C, or a bounded segment $[v, v']$ for some $v' \in E_R$ such that $v' \neq v$. The unbounded ray case is impossible, since

$$\omega(v + tu) = \omega(v) + t\omega(u)$$

would imply that ω is unbounded on C. So $E_R = [v, v']$. v' is a vertex of $[v, v']$, so by [28, Proposition 2.10], v' is a vertex of C. Since $[v, v']$ is a one-dimensional face of C, v' is a vertex neighboring v by definition.

Also since $v' \neq v$, $v' = v + tu$ for some $t > 0$, so

$$\omega(v') = \omega(v + tu) = \omega(v) + t\omega(u) > \omega(v).$$

□

7.3 Traversal formula

We now state the general version of the traversal formula of Theorem 4. Theorem 12 is the result of iteratively applying the jump formula through paths in the graph.

Theorem 12 (General traversal formula)**.** *(→ proof)*

1. *Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be a locally finite real affine hyperplane arrangement.*
2. *Let $\Omega(\mathcal{P})$ be the set of chambers as in Definition 11.*
3. *Let $\mathfrak{c}_0 \in \Omega(\mathcal{P})$ be a chamber.*
4. *Let $\omega \in (\mathbb{R}^k)^* \setminus \{0\}$.*
5. *Let $G_\omega(\mathcal{P})$ be the directed graph in Definition 26 whose vertices are $\Omega(\mathcal{P})$. Each vertex has finitely many edges.*
6. *Let $\lambda(\mathfrak{c}, \omega) \in \mathbb{R} \cup \{\infty\}$ be as in Definition 12.*
7. *Let $f \in \Theta(\mathbb{C}^k, \mathcal{P})$, where $\Theta(\mathbb{C}^k, \mathcal{P})$ is the $\mathbb{C}$-vector space of functions in Definition 46.*

For all $L \in \mathbb{Z}_{\geq 1}$, with $\mathbb{P}_j(\mathfrak{c}_0, G_\omega(\mathcal{P}))$ denoting the set of paths in $G_\omega(\mathcal{P})$ of length j starting at $\mathfrak{c}_0$, with $\mathbb{P}_0(\mathfrak{c}_0, G_\omega(\mathcal{P})) := \{(\mathfrak{c}_0)\}$, and with path signs as in Definition 27, one has

$$\begin{aligned} J(\mathfrak{c}_0, \omega) = &\sum_{j=0}^{L-1} \sum_{(\mathfrak{c}_0, \ldots, \mathfrak{c}_j) \in \mathbb{P}_j(\mathfrak{c}_0, G_\omega(\mathcal{P}))} \operatorname{sign}(\mathfrak{c}_0, \ldots, \mathfrak{c}_j) I(\mathfrak{c}_j, \omega) \\ &+ \sum_{(\mathfrak{c}_0, \ldots, \mathfrak{c}_L) \in \mathbb{P}_L(\mathfrak{c}_0, G_\omega(\mathcal{P}))} \operatorname{sign}(\mathfrak{c}_0, \ldots, \mathfrak{c}_L) J(\mathfrak{c}_L, \omega), \end{aligned}$$

where

$$I(\mathfrak{c}, \omega) := \begin{cases} \frac{1}{(2\pi i)^{\dim(F(\mathfrak{c},\omega))}} \int_{c + i \operatorname{lin}(F(\mathfrak{c},\omega))} \operatorname{Res}_{FL(\mathfrak{c},\omega)}(f \, \Pi_{\mathbb{C}^k}) \qquad c \in \operatorname{relint}(F(\mathfrak{c}, \omega)) & : \lambda(\mathfrak{c}, \omega) < \infty \\ 0 & : \lambda(\mathfrak{c}, \omega) = \infty \end{cases}$$

$$J(\mathfrak{c}, \omega) := \frac{1}{(2\pi i)^k} \int_{c + i\mathbb{R}^k} f \, \Pi_{\mathbb{C}^k} \qquad c \in \mathfrak{c}.$$

Here $\mathrm{Res}_{FL(\mathfrak{c},\omega)}$ *is the affine residue of Definition 42, and* I *and* J *are independent of* c.

Proof of Theorem 12. First we record the one-step identity. Let $\mathfrak{c} \in \Omega(\mathcal{P})$. If $\lambda(\mathfrak{c},\omega) = \infty$, then by Definition 26 the only outgoing edge from $\mathfrak{c}$ is the self-edge, whose sign is 1, and by definition $I(\mathfrak{c},\omega) = 0$. Thus in this case

$$J(\mathfrak{c},\omega) = I(\mathfrak{c},\omega) + \sum_{\substack{\mathfrak{b}\colon (\mathfrak{c},\mathfrak{b}) \\ \text{is an edge in } G_\omega(\mathcal{P})}} \operatorname{sign}(\mathfrak{c},\mathfrak{b}) J(\mathfrak{b},\omega).$$

Now suppose $\lambda(\mathfrak{c},\omega) < \infty$. Let

$$F := F(\mathfrak{c},\omega), \qquad r := \operatorname{codim}(F), \qquad FL := FL(\mathfrak{c},\omega).$$

Applying the global jump formula, Corollary 6, to the face F, the flag FL, and the form $\Pi_{\mathbb{C}^k}$, gives

$$\sum_{\alpha\in\{\pm 1\}^r} \frac{\operatorname{sign}(\alpha)}{(2\pi i)^r} \int_{c_\alpha + i\mathbb{R}^k} f\,\Pi_{\mathbb{C}^k} = \int_{c+i\,\mathrm{lin}(F)} \mathrm{Res}_{FL}(f\,\Pi_{\mathbb{C}^k}),$$

where $c_\alpha \in \mathfrak{b}_{FL}(\alpha)$ and $c \in \mathrm{relint}(F)$. Dividing by $(2\pi i)^{\dim(F)}$, and using $r + \dim(F) = k$, yields

$$\sum_{\alpha\in\{\pm 1\}^r} \operatorname{sign}(\alpha) J(\mathfrak{b}_{FL}(\alpha),\omega) = I(\mathfrak{c},\omega).$$

For $\alpha = 1^r$, we have $\mathfrak{b}_{FL}(\alpha) = \mathfrak{c}$ and $\operatorname{sign}(\alpha) = 1$. Hence

$$J(\mathfrak{c},\omega) = I(\mathfrak{c},\omega) + \sum_{\alpha\in\{\pm 1\}^r\setminus\{1^r\}} (-\operatorname{sign}(\alpha)) J(\mathfrak{b}_{FL}(\alpha),\omega).$$

By Definition 27, $-\operatorname{sign}(\alpha) = \operatorname{sign}(\mathfrak{c},\mathfrak{b}_{FL}(\alpha))$. Thus, in all cases,

$$J(\mathfrak{c},\omega) = I(\mathfrak{c},\omega) + \sum_{\substack{\mathfrak{b}\colon (\mathfrak{c},\mathfrak{b}) \\ \text{is an edge in } G_\omega(\mathcal{P})}} \operatorname{sign}(\mathfrak{c},\mathfrak{b}) J(\mathfrak{b},\omega). \tag{$*$}$$

The locally constant condition built into Definition 46 shows that $J(\mathfrak{c},\omega)$ is independent of the choice of $c \in \mathfrak{c}$. If $\lambda(\mathfrak{c},\omega) < \infty$, the identity above shows that $I(\mathfrak{c},\omega)$ is independent of the choice of $c \in \mathrm{relint}(F(\mathfrak{c},\omega))$, since it is expressed as a signed sum of the chamber quantities $J(\mathfrak{b}_{FL}(\alpha),\omega)$. If $\lambda(\mathfrak{c},\omega) = \infty$, this independence is immediate from the definition $I(\mathfrak{c},\omega) = 0$.

We now iterate $(*)$. For $L = 1$, $(*)$ at $\mathfrak{c}_0$ is exactly the desired formula, because $\mathbb{P}_0(\mathfrak{c}_0, G_\omega(\mathcal{P})) = \{(\mathfrak{c}_0)\}$, paths of length 1 are precisely outgoing edges from $\mathfrak{c}_0$, and $\operatorname{sign}(\mathfrak{c}_0) = 1$.

Assume the formula has been proved for a fixed $L \geq 1$. Then

$$\begin{aligned} J(\mathfrak{c}_0,\omega) = &\sum_{j=0}^{L-1} \sum_{(\mathfrak{c}_0,\ldots,\mathfrak{c}_j)} \operatorname{sign}(\mathfrak{c}_0,\ldots,\mathfrak{c}_j) I(\mathfrak{c}_j,\omega) \\ &+ \sum_{(\mathfrak{c}_0,\ldots,\mathfrak{c}_L)} \operatorname{sign}(\mathfrak{c}_0,\ldots,\mathfrak{c}_L) J(\mathfrak{c}_L,\omega), \end{aligned}$$

where the inner sum in the first term is over paths of length j starting at $\mathfrak{c}_0$, and the second sum is over paths of length L starting at $\mathfrak{c}_0$. Apply $(*)$ to each $J(\mathfrak{c}_L, \omega)$. Then

$$\sum_{(\mathfrak{c}_0,\ldots,\mathfrak{c}_L)} \operatorname{sign}(\mathfrak{c}_0,\ldots,\mathfrak{c}_L) J(\mathfrak{c}_L,\omega) = \sum_{(\mathfrak{c}_0,\ldots,\mathfrak{c}_L)} \operatorname{sign}(\mathfrak{c}_0,\ldots,\mathfrak{c}_L) I(\mathfrak{c}_L,\omega) + \sum_{(\mathfrak{c}_0,\ldots,\mathfrak{c}_{L+1})} \operatorname{sign}(\mathfrak{c}_0,\ldots,\mathfrak{c}_{L+1}) J(\mathfrak{c}_{L+1},\omega),$$

because path signs multiply along edges by Definition 27. Combining the I terms gives the claimed formula for paths of length $L+1$. This completes the induction. □

Proof of Theorem 4. By Proposition 14, the decay hypothesis in Theorem 4 implies that $f \in \Theta(\mathbb{C}^k, \mathcal{P})$. Therefore Theorem 12 applies, and its conclusions are exactly the stated formulas for $I(\mathfrak{c}, \omega)$ and $J(\mathfrak{c}, \omega)$. □

8 I and J

In Theorem 3, the expansion is written in terms of two families of quantities: remainder contour integrals $J_\sigma(\mathfrak{c}, \omega)$ and residue contributions $I_\sigma(\mathfrak{c}, \omega)$. This section provides the tools needed to bound $J_\sigma(\mathfrak{c}, \omega)$ and compute $I_\sigma(\mathfrak{c}, \omega)$.

8.1 Remainder terms

Proposition 15 is the bound on the remainder terms.

Proposition 15. *(→ proof) Let $V \subseteq \mathbb{R}^k$ be non-empty. Let $f : V + i\mathbb{R}^k \to \mathbb{C}$. Suppose that*

1. *For all $c \in V$, $y \to f(c + iy)$ is Lebesgue measurable.*
2. *For all $c \in V$, $\int_{c+i\mathbb{R}^k} |f|\,|\Pi_{\mathbb{C}^k}| < \infty$.*
3. *For all $c_1, c_2 \in V$ and $\phi \in (\mathbb{R}^k)^*$, $\int_{c_1+i\mathbb{R}^k} e^{-\phi} f\,\Pi_{\mathbb{C}^k} = \int_{c_2+i\mathbb{R}^k} e^{-\phi} f\,\Pi_{\mathbb{C}^k}$.*

Let $|\cdot|$ denote any norm on $(\mathbb{R}^k)^*$. Let $B = \{\psi \in (\mathbb{R}^k)^* : |\psi| = 1\}$. For every upper semicontinuous function $\eta : B \to \mathbb{R}$ such that

$$\eta(\psi) < \sup_{v \in V} \psi(v) \qquad \forall \psi \in B,$$

and for all $c \in V$,

$$\frac{1}{(2\pi i)^k} \int_{c+i\mathbb{R}^k} e^{-\phi} f\,\Pi_{\mathbb{C}^k} \in o\Big(e^{-|\phi|\eta(\frac{\phi}{|\phi|})}\Big),$$

uniformly as $|\phi| \to \infty$. In particular, for all $c \in V$, $\phi \in (\mathbb{R}^k)^* \setminus \{0\}$, and $\eta \in \mathbb{R}$ such that $\eta < \sup_{v \in V} \phi(v)$,

$$\frac{1}{(2\pi i)^k} \int_{c+i\mathbb{R}^k} e^{-\sigma\phi} f\,\Pi_{\mathbb{C}^k} \in o\Big(e^{-\sigma\eta}\Big),$$

as $\sigma \to \infty$.

Proof of Proposition 15. For each $c \in V$, define

$$U_c := \{\psi \in B : \eta(\psi) < \psi(c)\}.$$

Because η is upper semicontinuous and $\psi \mapsto \psi(c)$ is continuous, each U_c is open. By hypothesis, $\{U_c\}_{c \in V}$ covers B. Since B is compact, there exist $c_1, \ldots, c_n \in V$ such that $B = \bigcup_{j=1}^n U_{c_j}$. Write $\phi \in (\mathbb{R}^k)^* \setminus \{0\}$ as $\phi = |\phi|\psi$ for $\psi = \frac{\phi}{|\phi|} \in B$. Then for some $j_\phi \in [1, n]$,

$$\phi(c_{j_\phi}) > |\phi|\eta\Big(\frac{\phi}{|\phi|}\Big).$$

By the contour-independence hypothesis, we may evaluate the normalized integral on $c_j + i\mathbb{R}^k$. Factoring $e^{-\phi(c_j)}$ outside and writing $z = c_j + iy$ gives, for all $j \in [1, n]$,

$$\frac{1}{(2\pi i)^k} \int_{c+i\mathbb{R}^k} e^{-\phi} f\,\Pi_{\mathbb{C}^k} = \frac{e^{-\phi(c_j)}}{(2\pi)^k} \int_{\mathbb{R}^k} e^{-i\phi(y)} f(c_j + iy)dy.$$

By hypothesis, $\int_{c_j+i\mathbb{R}^k} |f|\,|\Pi_{\mathbb{C}^k}| < \infty$. So by the Riemann-Lebesgue lemma, $\int_{\mathbb{R}^k} e^{-i\phi(y)} f(c_j+iy)dy \to 0$ as $|\phi| \to \infty$. Since there are only finitely many j, we also have

$$\max_{j\in[1,n]} \Big| \int_{\mathbb{R}^k} e^{-i\phi(y)} f(c_j + iy)dy \Big| \to 0$$

as $|\phi| \to \infty$. Then

$$\begin{aligned}
\Big|\frac{1}{(2\pi i)^k} \int_{c+i\mathbb{R}^k} e^{-\phi} f\,\Pi_{\mathbb{C}^k}\Big| &= \frac{e^{-\phi(c_j)}}{(2\pi)^k} \Big| \int_{\mathbb{R}^k} e^{-i\phi(y)} f(c_j + iy)dy \Big| && \forall j \in [1,n] \\
&= \frac{e^{-\phi(c_{j_\phi})}}{(2\pi)^k} \Big| \int_{\mathbb{R}^k} e^{-i\phi(y)} f(c_{j_\phi} + iy)dy \Big| \\
&< \frac{e^{-|\phi|\eta(\frac{\phi}{|\phi|})}}{(2\pi)^k} \max_{j\in[1,n]} \Big| \int_{\mathbb{R}^k} e^{-i\phi(y)} f(c_j + iy)dy \Big| \\
&\in o\Big(e^{-|\phi|\eta(\frac{\phi}{|\phi|})}\Big).
\end{aligned}$$

Next we fix $\phi \in (\mathbb{R}^k)^* \setminus \{0\}$ and let $\eta \in \mathbb{R}$ such that $\eta < \sup_{c\in V} \phi(c)$. There exists $c_0 \in V$ such that $\phi(c_0) > \eta$. Then

$$\begin{aligned}
\Big|\frac{1}{(2\pi i)^k} \int_{c+i\mathbb{R}^k} e^{-\sigma\phi} f\,\Pi_{\mathbb{C}^k}\Big| &= \Big|\frac{1}{(2\pi i)^k} \int_{c_0+i\mathbb{R}^k} e^{-\sigma\phi} f\,\Pi_{\mathbb{C}^k}\Big| \\
&= \frac{e^{-\sigma\phi(c_0)}}{(2\pi)^k} \Big| \int_{\mathbb{R}^k} e^{-i\sigma\phi(y)} f(c_0 + iy)dy \Big| \\
&< \frac{e^{-\sigma\eta}}{(2\pi)^k} \Big| \int_{\mathbb{R}^k} e^{-i\sigma\phi(y)} f(c_0 + iy)dy \Big|.
\end{aligned}$$

This is $o(e^{-\sigma\eta})$ by the Riemann-Lebesgue lemma. □

8.2 Coefficient extraction with derivatives

We now return to the residue terms. The goal is to develop a computational formula for the residue in terms of derivatives, so that the coefficients in the asymptotic expansion can be read off explicitly. Our formula can be viewed as a generalization of the classical formula that for a meromorphic function of one variable $f \in \mathcal{M}_a$ such that $f = \frac{g}{(z-a)^d}$ for some g that is holomorphic on an open neighborhood of a,

$$\mathrm{Res}_{z=a} f = \frac{1}{(d-1)!} \partial_z^{d-1} g\big|_{z=a}.$$

Proof of Lemma 5. Set $g := \gamma h$. $\mathrm{div}(h) = \mathrm{div}(\gamma)_-$, so $\mathrm{div}(g) = \mathrm{div}(\gamma h) = \mathrm{div}(\gamma) + \mathrm{div}(h) = \mathrm{div}(\gamma)_+ - \mathrm{div}(\gamma)_- + \mathrm{div}(\gamma)_- = \mathrm{div}(\gamma)_+ \geq 0$. So $g \in \mathcal{O}(U)$.

Suppose towards contradiction that g vanishes identically on H_i. Then $\mathrm{div}(\gamma)_+$ and $\mathrm{div}(\gamma)_-$ both have positive coefficients for H_i, which is a contradiction. □

Proof of Proposition 7. Let

$$\mathrm{div}(\gamma)_- = \sum_{i=1}^n e_i H_i,$$

for $e_1, \dots, e_n \in \mathbb{Z}_{>0}$ and complex affine hyperplanes $H_1, \dots, H_n$ from $\mathcal{P}_{\mathbb{C}}$ that intersect U. Choose $q_1, \dots, q_n \in \mathfrak{Q}(\mathcal{P})$ such that $Z_{\mathbb{C}}(q_i) = H_i$. We claim that each q_i lies in $\mathfrak{Q}(F, \mathcal{P})$. If not, then $q_i \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})$, so the hypothesis on U gives $U \cap Z_{\mathbb{C}}(q_i) = \emptyset$. But $Z_{\mathbb{C}}(q_i) = H_i$, and H_i was chosen to intersect U, a contradiction. Let $h := \prod_{i=1}^{n} q_i^{e_i}$. By Lemma 5, there exists $g \in \mathcal{O}(U)$ such that

$$\gamma = \frac{g}{h}$$

on U. Pulling back through b:

$$\widetilde{\gamma}(x, y) = \frac{g \circ b(x, y)}{\displaystyle\prod_{i=1}^{n} \left(q_i \circ b(x, y)\right)^{e_i}}.$$

Since $b(0, y) = y$ and g is holomorphic near $\operatorname{relint}(F) + i \operatorname{lin}(F)$, $g \circ b$ is holomorphic near $\{0\} \times (\operatorname{relint}(F) + i \operatorname{lin}(F))$.

Now consider each $q \in \mathfrak{Q}(F, \mathcal{P})$. By Corollary 4, using the affine coordinates from Definition 28, we have $q = \sum_{i=1}^{\operatorname{gi}(q)} a_i(q) \rho_i^{FL}$. Again pulling back through b,

$$u_i^*(u(x)) = \prod_{j=i}^{r} x_j^*(x).$$

For $y \in F_{\mathbb{C}}$, $\rho_i^{FL}(y) = 0$, so

$$q \circ b(x, y) = \sum_{i=1}^{\operatorname{gi}(q)} a_i(q) \prod_{j=i}^{r} x_j^*(x).$$

Factoring out $\operatorname{gc}(q) \prod_{j=\operatorname{gi}(q)}^{r} x_j^*(x)$:

$$q \circ b(x, y) = \operatorname{gc}(q) \prod_{j=\operatorname{gi}(q)}^{r} x_j^*(x) \cdot \left(1 + \sum_{i=1}^{\operatorname{gi}(q)-1} \frac{a_i(q)}{\operatorname{gc}(q)} \prod_{j=i}^{\operatorname{gi}(q)-1} x_j^*(x)\right).$$

The factor in parentheses equals 1 at $x = 0$, so it is an analytic unit in an open neighborhood of $\{0\} \times (\operatorname{relint}(F) + i \operatorname{lin}(F))$. Therefore

$$\left(q_i \circ b(x, y)\right)^{-e_i} = \frac{1}{\operatorname{gc}(q_i)^{e_i}} \cdot \Big(\prod_{j=\operatorname{gi}(q_i)}^{r} x_j^*(x) \Big)^{-e_i} \cdot (\text{analytic unit}).$$

So the pole order of $(q_i \circ b)^{-e_i}$ in x_j^* is

$$e_i \mathbf{1}_{\operatorname{gi}(q_i) \le j}.$$

Taking the product over $1, \dots, n$, the pole order of $\prod_{i=1}^{n} (q_i \circ b)^{-e_i}$ in x_j^* is

$$\sum_{\substack{i \in [1,n] \\ \operatorname{gi}(q_i) \le j}} e_i = \sum_{i=1}^{n} e_i \, \mathbf{1}_{\operatorname{gi}(q_i) \le j} = d_j.$$

Therefore, multiplying by $\prod_{i=1}^r (x_i^*)^{d_i}$ exactly cancels all negative powers. Explicitly,

$$\begin{aligned}\Big(\prod_{i=1}^r (x_i^*)^{d_i}\Big)\widetilde{\gamma} &= (g\circ b)\cdot \prod_{i=1}^n \frac{\big(\prod_{j=\mathrm{gi}(q_i)}^r x_j^*\big)^{e_i}}{\big(\,\mathrm{gc}(q_i)\prod_{j=\mathrm{gi}(q_i)}^r x_j^*\big)^{e_i}}\cdot(\text{analytic unit})\\ &= (g\circ b)\cdot(\text{analytic unit}).\end{aligned}$$

Since $g\circ b$ is holomorphic near $\{0\}\times(\mathrm{relint}(F)+i\,\mathrm{lin}(F))$, the product $\big(\prod_{i=1}^r (x_i^*)^{d_i}\big)\widetilde{\gamma}$ analytically continues to an open neighborhood of $\{0\}\times(\mathrm{relint}(F)+i\,\mathrm{lin}(F))$. □

We also have the following coefficient extraction formula.

Proposition 16. *(→ proof) Assume the setting of Definition 39. Let $\gamma\in R(U,\mathcal{P})$ have a representation $\gamma=\frac{g}{\prod_{i=1}^n q_i^{e_i}}$ as in Lemma 5. Let d_j and D_j be as in Definition 29, for this F, FL, and representation, and let $\widetilde{\gamma}$ be as in Proposition 7. Let $\rho_i=\rho_i^{FL}$ be the canonical affine coordinates from Corollary 4, let $\eta_H:=d\rho_1\wedge\cdots\wedge d\rho_r$, and let $\lambda_F\in\Lambda^{k-r}\,\mathrm{lin}(F)^*_{\mathbb{C}}$. Then*

$$\mathrm{Res}_{FL}(\gamma\,\eta_H\wedge\lambda_F)=\frac{1}{\prod_{j=1}^r (D_j-j)!}\big[(\prod_{j=1}^r \partial_{x_j}^{D_j-j})(\prod_{j=1}^r (x_j^*)^{D_j}\widetilde{\gamma})\big]\big|_{x=0}\lambda_F.$$

Proof of Proposition 16. $\mathrm{Res}_{FL}(\gamma\,\eta_H\wedge\lambda_F)$ extracts the coefficient $a_{-1,\dots,-1}$ from the Laurent series $\gamma=\sum_{\beta\in\mathbb{Z}^r} a_\beta\rho^\beta$. Since $\rho_i\circ b=\prod_{j=i}^r x_j^*$, this corresponds to the coefficient $b_{-1,-2,\dots,-r}$ from the Laurent series $\widetilde{\gamma}=\sum_{\beta\in\mathbb{Z}^r} b_\beta(x^*)^\beta$. Because $\prod_{j=1}^r (x_j^*)^{d_j}\widetilde{\gamma}$ is locally holomorphic by Proposition 7 and $D_j\ge d_j$, the product $\prod_{j=1}^r (x_j^*)^{D_j}\widetilde{\gamma}$ is also locally holomorphic. The coefficient $b_{-1,-2,\dots,-r}$ is the coefficient of $\prod_{j=1}^r (x_j^*)^{D_j-j}$ in this holomorphic product. The displayed derivative formula extracts that coefficient. □

Corollary 8. *(→ proof) Assume the setting of Proposition 16. Let $\phi\in(\mathbb{R}^k)^*$. Let $a_1,\dots,a_r\in\mathbb{R}$ and let $\phi_F:F_{\mathbb{C}}\to\mathbb{C}$ be the affine restriction of ϕ such that, for all $u\in(\mathrm{lin}(F)^\perp)_{\mathbb{C}}$ and $y\in F_{\mathbb{C}}$,*

$$\phi(u+y)=\sum_{i=1}^r a_i u_i^*(u)+\phi_F(y).$$

Then

$$\begin{aligned}\mathrm{Res}_{FL}(e^{-\phi}\gamma\,\eta_H\wedge\lambda_F)=e^{-\phi_F}\sum_\beta\Big[\prod_{i=1}^r \frac{(-a_i)^{\beta_i-\beta_{i-1}}}{(\beta_i-\beta_{i-1})!(D_i-i-\beta_i)!}\\ \Big[\prod_{i=1}^r \partial_{x_i}^{D_i-i-\beta_i}\Big](\prod_{i=1}^r (x_i^*)^{D_i}\widetilde{\gamma})\Big]\Big|_{x=0}\lambda_F,\end{aligned}$$

where β ranges over $\mathbb{Z}^r_{\ge 0}$ such that $\beta_i\le D_i-i$ and $\beta_1\le\cdots\le\beta_r$ and with the convention $\beta_0=0$.

Proof of Corollary 8. Since $b(x,y)=u(x)+y$ and

$$u_i^*(u(x))=\prod_{j=i}^r x_j^*,$$

the decomposition

$$\phi(u+y) = \sum_{i=1}^{r} a_i u_i^*(u) + \phi_F(y)$$

gives

$$\phi \circ b(x,y) = \sum_{i=1}^{r} a_i \prod_{j=i}^{r} x_j^*(x) + \phi_F(y).$$

Therefore,

$$e^{-\phi \circ b(x,y)} = e^{-\phi_F(y)} e^{-\sum_{i=1}^{r} a_i \prod_{j=i}^{r} x_j^*(x)}.$$

Next,

$$\begin{aligned} e^{-\sum_{i=1}^{r} a_i \prod_{j=i}^{r} x_j^*} &= \sum_{\alpha \in \mathbb{Z}_{\geq 0}^r} \frac{\prod_{i=1}^{r} (-a_i)^{\alpha_i}}{\prod_{i=1}^{r} \alpha_i!} \prod_{i=1}^{r} (\prod_{j=i}^{r} x_j^*)^{\alpha_i} \\ &= \sum_{\alpha \in \mathbb{Z}_{\geq 0}^r} \frac{\prod_{i=1}^{r} (-a_i)^{\alpha_i}}{\prod_{i=1}^{r} \alpha_i!} \prod_{i=1}^{r} (x_i^*)^{\sum_{j=1}^{i} \alpha_j}. \end{aligned}$$

We extract the coefficient for $\prod_{i=1}^{r} (x_i^*)^{\beta_i}$. The unique solution α to

$$\prod_{i=1}^{r} (x_i^*)^{\sum_{j=1}^{i} \alpha_j} = \prod_{i=1}^{r} (x_i^*)^{\beta_i}$$

is $\alpha_i = \beta_i - \beta_{i-1}$, with the convention $\beta_0 = 0$. So the coefficient for $\prod_{i=1}^{r} (x_i^*)^{\beta_i}$ is

$$\begin{cases} \prod_{i=1}^{r} \frac{(-a_i)^{\beta_i - \beta_{i-1}}}{(\beta_i - \beta_{i-1})!} & : \beta_i \geq \beta_{i-1} \forall i \in [1,r] \\ 0 & : \text{otherwise} \end{cases},$$

and therefore

$$[\prod_{i=1}^{r} \partial_{x_i}^{\beta_i}](e^{-\phi \circ b})\Big|_{x=0} = \begin{cases} e^{-\phi_F} \prod_{i=1}^{r} \frac{\beta_i! (-a_i)^{\beta_i - \beta_{i-1}}}{(\beta_i - \beta_{i-1})!} & : \beta_i \geq \beta_{i-1} \forall i \in [1,r] \\ 0 & : \text{otherwise.} \end{cases}$$

By Proposition 16,

$$\text{Res}_{FL}(e^{-\phi} \gamma \, \eta_H \wedge \lambda_F) = \frac{1}{\prod_{i=1}^{r} (D_i - i)!} [[\prod_{i=1}^{r} \partial_{x_i}^{D_i - i}](e^{-\phi \circ b} \prod_{i=1}^{r} (x_i^*)^{D_i} \widetilde{\gamma})]\big|_{x=0} \lambda_F.$$

Next, we apply the product rule:

$$\begin{aligned}
&\frac{1}{\prod_{i=1}^r (D_i-i)!}[[\prod_{i=1}^r \partial_{x_i}^{D_i-i}](e^{-\phi\circ b}\prod_{i=1}^r (x_i^*)^{D_i}\widetilde{\gamma})]\big|_{x=0}\lambda_F \\
&= \frac{1}{\prod_{i=1}^r (D_i-i)!}\sum_{\beta}[\prod_{i=1}^r \binom{D_i-i}{\beta_i}[\prod_{i=1}^r \partial_{x_i}^{\beta_i}](e^{-\phi\circ b})[\prod_{i=1}^r \partial_{x_i}^{D_i-i-\beta_i}](\prod_{i=1}^r (x_i^*)^{D_i}\widetilde{\gamma})]\big|_{x=0}\lambda_F \\
&= \frac{e^{-\phi_F}}{\prod_{i=1}^r (D_i-i)!}\sum_{\substack{\beta \\ \beta \text{ non-decreasing}}}[\prod_{i=1}^r \binom{D_i-i}{\beta_i}\prod_{i=1}^r \frac{\beta_i!(-a_i)^{\beta_i-\beta_{i-1}}}{(\beta_i-\beta_{i-1})!}[\prod_{i=1}^r \partial_{x_i}^{D_i-i-\beta_i}](\prod_{i=1}^r (x_i^*)^{D_i}\widetilde{\gamma})]\big|_{x=0}\lambda_F \\
&= e^{-\phi_F}\sum_{\substack{\beta \\ \beta \text{ non-decreasing}}}[\prod_{i=1}^r \frac{(-a_i)^{\beta_i-\beta_{i-1}}}{(\beta_i-\beta_{i-1})!(D_i-i-\beta_i)!}[\prod_{i=1}^r \partial_{x_i}^{D_i-i-\beta_i}](\prod_{i=1}^r (x_i^*)^{D_i}\widetilde{\gamma})]\big|_{x=0}\lambda_F,
\end{aligned}$$

where β ranges over $\mathbb{Z}^r_{\geq 0}$ such that $\beta \leq (D_1-1,\ldots,D_r-r)$ and β is non-decreasing. □

Proof of Proposition 8. Let $u \in \operatorname{lin}(F(\mathfrak{c},\omega))^\perp$ and $y \in F(\mathfrak{c},\omega)_{\mathbb{C}}$. ω is linear, so $\omega(u+y)=\omega(u)+\omega(y)$. Since

$$F(\mathfrak{c},\omega)_{\mathbb{C}} = \operatorname{aff}(F(\mathfrak{c},\omega)) + i\operatorname{lin}(F(\mathfrak{c},\omega)),$$

we can write $y = x+iv$ with $x \in F(\mathfrak{c},\omega)$ and $v \in \operatorname{lin}(F(\mathfrak{c},\omega))$. By definition of $F(\mathfrak{c},\omega)$, $\omega(x)=\lambda(\mathfrak{c},\omega)$. Also, because ω is constant on the face $F(\mathfrak{c},\omega)$, its linear part vanishes on $\operatorname{lin}(F(\mathfrak{c},\omega))$, so $\omega(v)=0$. Therefore

$$\omega(y) = \omega(x+iv) = \omega(x) + i\omega(v) = \lambda(\mathfrak{c},\omega).$$

To evaluate $\omega(u)$, we restrict to $\omega\big|_{\operatorname{lin}(F(\mathfrak{c},\omega))^\perp} \in (\operatorname{lin}(F(\mathfrak{c},\omega))^\perp)^*$. Since $u_1^*,\ldots,u_r^*$ is a basis for $(\operatorname{lin}(F(\mathfrak{c},\omega))^\perp)^*$, there exists $a_1,\ldots,a_r \in \mathbb{R}$ such that $\omega\big|_{\operatorname{lin}(F(\mathfrak{c},\omega))^\perp}(u) = \sum_{i=1}^r a_i u_i^*(u)$ for all $u \in \operatorname{lin}(F(\mathfrak{c},\omega))^\perp$. Thus

$$\omega(u+y) = \omega(u)+\omega(y) = \omega\big|_{\operatorname{lin}(F(\mathfrak{c},\omega))^\perp}(u) + \omega(y) = \sum_{i=1}^r a_i u_i^*(u) + \lambda(\mathfrak{c},\omega)$$

Next we show that $a_j = \mathfrak{P}(F_j(\mathfrak{c},\omega),\mathfrak{c},\omega)$. By definition of the dual basis, $u_i^*(u_j)=\delta_{ij}$, so

$$\omega(u_j) = \sum_{i=1}^r a_i u_i^*(u_j) = a_j$$

By Definition 19, u_j is the generator of $F_{j-1}(\mathfrak{c},\omega)$ over $F_j(\mathfrak{c},\omega)$. By Definition 18, $F_{j-1}(\mathfrak{c},\omega)$ is an optimal face for $\mathfrak{P}(F_j(\mathfrak{c},\omega),\mathfrak{c},\omega)$, as in Definition 16. So

$$\mathfrak{P}(F_j(\mathfrak{c},\omega),\mathfrak{c},\omega) = \omega(F_{j-1}(\mathfrak{c},\omega),F_j(\mathfrak{c},\omega)) = \omega(u_j)$$

Thus, $a_j = \omega(u_j) = \mathfrak{P}(F_j(\mathfrak{c},\omega),\mathfrak{c},\omega)$. By Lemma 39, $\mathfrak{P}(F_j(\mathfrak{c},\omega),\mathfrak{c},\omega) < 0$. □

Proof of Lemma 6. Suppose $x_1, x_2 \in \operatorname{relint}(F)$. Then $\operatorname{sign}(q(x_1)) = \operatorname{sign}_F(q) = \operatorname{sign}(q(x_2))$ for all $q \in \mathfrak{Q}(\mathcal{P})$. In particular, $\operatorname{sign}(q(x_1)) = \operatorname{sign}(q(x_2))$ for all $q \in \mathfrak{Q}(\widetilde{\mathcal{P}})$. So x_1 and x_2 are in the same chamber in $\Omega(\widetilde{\mathcal{P}})$. Hence V_x is independent of the choice of $x \in \operatorname{relint}(F)$, and V_F is well-defined.

For every $x \in \operatorname{relint}(F)$, we have $x \in V_x = V_F$, so $\operatorname{relint}(F) \subseteq V_F$. Since F has positive codimension, $\operatorname{relint}(F)$ is contained in the proper affine subspace $\operatorname{aff}(F) \subsetneq \mathbb{R}^k$, whereas V_F is a nonempty open subset of $\mathbb{R}^k$. Therefore $\operatorname{relint}(F) \subsetneq V_F$.

Since V_F is a chamber of $\widetilde{\mathcal{P}}$, it is disjoint from $\widetilde{\mathcal{P}}$. Equivalently,

$$V_F \cap Z(q) = \emptyset \qquad q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P}).$$

Since V_F is a chamber of a real affine hyperplane arrangement, it is convex. Thus for every $q \in \mathfrak{Q}(F, \mathcal{P})$, the set $V_F \cap Z(q)$ is convex and nonempty because $\operatorname{relint}(F) \subseteq V_F \cap Z(q)$. Hence $V_F \cap Z(q)$ is connected.

Since F has positive codimension, $\operatorname{relint}(F) + i \operatorname{lin}(F)$ is contained in the proper complex affine subspace $\operatorname{aff}(F) + i \operatorname{lin}(F) \subsetneq \mathbb{C}^k$, whereas U_F is a nonempty open subset of $\mathbb{C}^k$. So $\operatorname{relint}(F) + i \operatorname{lin}(F) \subsetneq U_F$. Finally, let $z = x + iy \in U_F$ with $x \in V_F$ and $y \in \mathbb{R}^k$. If $q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})$, then $q(x) \neq 0$ because we showed $V_F \cap Z(q) = \emptyset$. Then $\operatorname{Re}(q(z)) = q(\operatorname{Re}(z)) = q(x) \neq 0$, so $q(z) \neq 0$. Hence $U_F \cap Z_{\mathbb{C}}(q) = \emptyset$ for every $q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})$.

Now let $q \in \mathfrak{Q}(F, \mathcal{P})$. By the nonemptiness and connectedness statement for V_F,

$$U_F \cap Z_{\mathbb{C}}(q) = (V_F \cap Z(q)) + i \ker(dq)$$

is nonempty and connected. □

Proof of Lemma 7. Choose $x \in \operatorname{relint}(F)$. Since $F \subseteq \overline{\mathfrak{c}}$, we have $x \in \overline{\mathfrak{c}}$. Let

$$\widetilde{\mathcal{P}} := \bigcup_{q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})} Z(q)$$

as in Lemma 6. If $q \in \mathfrak{Q}(\mathcal{P}) \setminus \mathfrak{Q}(F, \mathcal{P})$, then $q(x) \neq 0$. The sign of q is constant on the chamber $\mathfrak{c}$, and because $x \in \overline{\mathfrak{c}}$, this sign agrees with $\operatorname{sign}(q(x))$. Thus every point of $\mathfrak{c}$ has the same signs as x with respect to the arrangement $\widetilde{\mathcal{P}}$. Hence $\mathfrak{c}$ lies in the chamber V_F of $\widetilde{\mathcal{P}}$ containing x.

If $\mathcal{P}$ has at least one positive codimension face, then every chamber has a boundary face of positive codimension in its closure. Applying the first part to such a face gives the final statement. □

Proof of Lemma 8. Because $\mathcal{P}$ is locally finite, $\mathfrak{Q}(F, \mathcal{P})$ is finite, up to scaling. So $\bigcup_{q \in \mathfrak{Q}(F, \mathcal{P})} Z_{\mathbb{C}}(q)$ is a union of finitely many complex affine hyperplanes, $H_1, \ldots, H_n$. Choose $q_1, \ldots, q_n \in \mathfrak{Q}(F, \mathcal{P})$ such that $Z_{\mathbb{C}}(q_i) = H_i$.

Let $\gamma \in \widehat{R}(U, \mathcal{P})$. Since the hyperplanes not containing F are disjoint from U, every pole of γ in U lies on one of the hyperplanes $H_1, \ldots, H_n$. Thus there are integers $e_1, \ldots, e_n \in \mathbb{Z}_{\geq 0}$ such that $\prod_{i=1}^n q_i^{e_i}$ clears all poles of γ on U. Indeed, for each i, the hypothesis gives finitely many connected components of $U \cap H_i$. On each such component, γ has a finite pole order on H_i because $\gamma \in \widehat{R}(U, \mathcal{P})$, and the pole order is locally constant along the smooth hypersurface H_i. Since there are finitely

many connected components, there is a finite maximum pole order along $U \cap H_i$; take e_i to be this maximum. Set $h := \prod_{i=1}^{n} q_i^{e_i}$ and $g := \gamma h$. Then

$$\operatorname{div}(g) = \operatorname{div}(\gamma h) = \operatorname{div}(\gamma) + \operatorname{div}(h) \geq 0.$$

Hence $g \in \mathcal{O}(U)$ and

$$\gamma = \frac{g}{h}.$$

Therefore $\gamma \in R(U, \mathcal{P})$. □

9 Combining results for the main theorems

In this section, we complete the proof of the main theorems.

9.1 Contour representation of the Laplace integral

We will use the following consequence of Mellin inversion to write M_σ as a contour integral, which is the starting point for the contour shifting argument.

Proposition 17. *(→ proof) Let $c \in \mathbb{R}^k$. Let μ be a finite signed Radon measure on $(0,\infty)^k$ and let $|\mu|$ denote its total variation. Suppose that*

$$\int_{(0,\infty)^k} x^c d|\mu| < \infty.$$

Let $g : (0,\infty)^k \to \mathbb{C}$ be such that $h : \mathbb{R}^k \to \mathbb{C}$ defined by $h(\tau) := e^{-\langle c,\tau\rangle} g(e^\tau)$ is Schwartz. Let $e_1^*, \dots, e_k^*$ be the standard dual basis on $\mathbb{C}^k$. Let $\Pi_{\mathbb{C}^k} := \bigwedge_{i=1}^k de_i^*$ be the standard holomorphic volume form on $\mathbb{C}^k$, restricted to the contour $c + i\mathbb{R}^k$ in the integral below. Let $d^\times x$ be the multiplicative Haar measure on $(0,\infty)^k$.

Define for $z \in c + i\mathbb{R}^k$

$$\zeta(z) := \int_{(0,\infty)^k} x^z d\mu(x)$$

and

$$\kappa(z) = \int_{(0,\infty)^k} x^{-z} g(x) d^\times x$$

Then

$$\int_{(0,\infty)^k} g d\mu = \frac{1}{(2\pi i)^k} \int_{c+i\mathbb{R}^k} \zeta\kappa\, \Pi_{\mathbb{C}^k}$$

Proof of Proposition 17. Let $\widetilde{\mu}$ be the finite signed Radon measure on $(0,\infty)^k$ defined by

$$d\widetilde{\mu}(x) := x^c\, d\mu(x).$$

Let ν be the push-forward of $\widetilde{\mu}$ through the map $(\log x_1, \dots, \log x_k)$. Then ν is a finite signed Radon measure and

$$|\nu|(\mathbb{R}^k) \le |\widetilde{\mu}|((0,\infty)^k) = \int_{(0,\infty)^k} x^c\, d|\mu| < \infty.$$

Moreover,

$$\begin{aligned}\int_{(0,\infty)^k} g\, d\mu &= \int_{(0,\infty)^k} (x^{-c} g)\, d\widetilde{\mu} \\ &= \int_{\mathbb{R}^k} h\, d\nu.\end{aligned}$$

Let $\mathcal{F}$ be the Fourier transform with convention

$$\mathcal{F}[\varphi](\xi) := \int e^{-i\langle \tau,\xi\rangle} \varphi(\tau)\, d\tau.$$

Because h is Schwartz, $\mathcal{F}^{-1}[\mathcal{F}[h]] = h$ and $\mathcal{F}[h] \in L^1(\mathbb{R}^k)$.

$$\begin{aligned}\int_{\mathbb{R}^k} h\, d\nu &= \int_{\mathbb{R}^k} \mathcal{F}^{-1}[\mathcal{F}[h]]\, d\nu \\ &= \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} \left[\int_{\mathbb{R}^k} e^{i\langle \tau, \xi \rangle} \mathcal{F}[h](\xi)\, d\xi \right] d\nu(\tau).\end{aligned}$$

$e^{i\langle \tau, \xi \rangle} \mathcal{F}[h](\xi)$ is integrable with respect to $d\xi \times d|\nu|(\tau)$ because $\mathcal{F}[h] \in L^1(\mathbb{R}^k)$ and $|\nu|(\mathbb{R}^k) < \infty$. This justifies Fubini's theorem.

$$\begin{aligned}\frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} \left[\int_{\mathbb{R}^k} e^{i\langle \tau, \xi \rangle} \mathcal{F}[h](\xi)\, d\xi \right] d\nu(\tau) &= \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} \left[\int_{\mathbb{R}^k} e^{i\langle \tau, \xi \rangle}\, d\nu(\tau) \right] \mathcal{F}[h](\xi)\, d\xi \\ &= \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} \left[\int_{(0,\infty)^k} e^{i\langle \log x, \xi \rangle}\, d\widetilde{\mu}(x) \right] \mathcal{F}[h](\xi)\, d\xi \\ &= \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} \left[\int_{(0,\infty)^k} x^{c+i\xi}\, d\mu(x) \right] \mathcal{F}[h](\xi)\, d\xi \\ &= \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} \zeta(c + i\xi) \mathcal{F}[h](\xi)\, d\xi \\ &= \frac{1}{(2\pi i)^k} \int_{c+i\mathbb{R}^k} \zeta(z) \mathcal{F}[h](-i(z-c))\, \Pi_{\mathbb{C}^k}.\end{aligned}$$

Next,

$$\begin{aligned}\mathcal{F}[h](\xi) &= \int_{\mathbb{R}^k} e^{-i\langle \tau, \xi \rangle} h(\tau)\, d\tau \\ &= \int_{(0,\infty)^k} x^{-(c+i\xi)} g(x)\, d^{\times}x.\end{aligned}$$

Since $c + i(-i(z-c)) = z$,

$$\mathcal{F}[h](-i(z-c)) = \int_{(0,\infty)^k} x^{-z} g(x)\, d^{\times}x = \kappa(z).$$

Chaining the results together,

$$\int_{(0,\infty)^k} g\, d\mu = \frac{1}{(2\pi i)^k} \int_{c+i\mathbb{R}^k} \zeta(z)\kappa(z)\, \Pi_{\mathbb{C}^k}.$$

□

We would like to apply Proposition 17 to $g(x) := e^{-\sum_{i=1}^k t_{\sigma i} x_i}$ and μ. We list the related functions, with h from Proposition 17 having explicit dependence on c:

$$\begin{aligned}h_c(\tau) &:= e^{-\langle c, \tau \rangle - \sum_{i=1}^k t_{\sigma i} e^{\tau_i}} \\ \kappa(z) &:= \int_{(0,\infty)^k} x^{-z} g(x) d^{\times}x \\ \zeta(z) &:= \int_{(0,\infty)^k} x^z d\mu(x)\end{aligned}$$

Lemma 42. *(→ proof) For $Re(z_i) < 0$ for all $i \in [1, k]$,*

$$\kappa(z) = \prod_{i=1}^{k} t_{\sigma i}^{z_i} \Gamma(-z_i).$$

Thus κ has the meromorphic continuation given by the right-hand side.

Proof of Lemma 42. Assume $\mathrm{Re}(z_i) < 0$ for all i. Then the following integrals converge absolutely, and

$$\begin{aligned}
\kappa(z) &= \int_{(0,\infty)^k} x^{-z} g(x) d^\times x \\
&= \int_{(0,\infty)^k} e^{-\sum_{i=1}^k t_{\sigma i} x_i} x^{-z} d^\times x \\
&= \int_{(0,\infty)^k} e^{-\sum_{i=1}^k x_i} \prod_{i=1}^{k} (\frac{x_i}{t_{\sigma i}})^{-z_i} d^\times x \\
&= \prod_{i=1}^{k} t_{\sigma i}^{z_i} \Gamma(-z_i).
\end{aligned}$$

□

Lemma 43. *(→ proof) Assume $c \in (-\infty, 0)^k$. Then h_c is Schwartz.*

Proof of Lemma 43. Because $c \in (-\infty, 0)^k$, $e^{-\langle c, \tau \rangle}$ and its derivatives vanish exponentially as $\tau_i \to -\infty$. Meanwhile, $e^{-\sum_{i=1}^k t_{\sigma i} e^{\tau_i}}$ and its derivatives vanish super-exponentially as $\tau_i \to \infty$. □

Proof of Lemma 2. For each $i \in [1, k]$, the analytic factor $\widetilde{f_i}$ is not identically zero on any connected component of some open neighborhood of W, by the main setting assumptions. The zero set of f_i is the same as the zero set of $\widetilde{f_i}$. Thus, by Lemma 1, $W \cap \bigcup_{i=1}^k f_i^{-1}(0)$ has Lebesgue measure 0.

Let

$$\nu := (f_1, \ldots, f_k)_\#(1_W |\phi(w)|\, dw).$$

Since μ is the push-forward of the signed measure $1_W(w)\phi(w)\, dw$ through $(f_1, \ldots, f_k)$, its total variation satisfies

$$|\mu| \leq \nu.$$

Set

$$A := \{x \in \mathbb{R}^k : x_1 \cdots x_k = 0\}, \qquad B := \mathbb{R}^k \setminus [0, \infty)^k.$$

Then

$$W \cap (f_1, \ldots, f_k)^{-1}(A) = W \cap \bigcup_{i=1}^{k} f_i^{-1}(0),$$

so $\nu(A) = 0$ and hence $|\mu|(A) = 0$. Also, each f_i is non-negative on W, so $W \cap (f_1, \ldots, f_k)^{-1}(B) = \emptyset$, so $\nu(B) = 0$ and therefore $|\mu|(B) = 0$. Since $\mathbb{R}^k \setminus (0,\infty)^k = A \cup B$, it follows that $|\mu|(\mathbb{R}^k \setminus (0,\infty)^k) = 0$. So,

$$\begin{aligned}\int_{(0,\infty)^k} e^{-\sum_{i=1}^k t_{\sigma i} x_i} d\mu(x) &= \int_{\mathbb{R}^k} e^{-\sum_{i=1}^k t_{\sigma i} x_i} d\mu(x) \\ &= \int_W e^{-\sum_{i=1}^k t_{\sigma i} f_i(w)} \phi(w)\, dw \\ &= M_\sigma.\end{aligned}$$

□

Now we derive the contour integral representation.

Proof of Proposition 1. By Lemma 3, there exists $c_0 \in (-\infty, 0)^k$ such that

$$\int_{(0,\infty)^k} x^{c_0}\, d|\mu|(x) < \infty.$$

Since $c_0 \in (-\infty, 0)^k$, Lemma 43 shows that h_{c_0} is Schwartz. Apply Proposition 17 with $c = c_0$ and $g(x) := e^{-\sum_{i=1}^k t_{\sigma i} x_i}$. Then

$$\int_{(0,\infty)^k} e^{-\sum_{i=1}^k t_{\sigma i} x_i}\, d\mu(x) = \frac{1}{(2\pi i)^k} \int_{c_0 + i\mathbb{R}^k} \zeta(z)\kappa(z)\, \Pi_{\mathbb{C}^k}.$$

Thus,

$$\begin{aligned}M_\sigma &= \int_W e^{-\sum_{i=1}^k t_{\sigma i} f_i(w)} \phi(w)\, dw \\ &= \int_{(0,\infty)^k} e^{-\sum_{i=1}^k t_{\sigma i} x_i}\, d\mu(x) && \text{Lemma 2} \\ &= \frac{1}{(2\pi i)^k} \int_{c_0 + i\mathbb{R}^k} \zeta(z)\kappa(z)\, \Pi_{\mathbb{C}^k} \\ &= \frac{1}{(2\pi i)^k} \int_{c_0 + i\mathbb{R}^k} \zeta(z) \prod_{i=1}^k t_{\sigma i}^{z_i} \Gamma(-z_i)\, \Pi_{\mathbb{C}^k} \\ &= \frac{1}{(2\pi i)^k} \int_{c_0 + i\mathbb{R}^k} e^{-\omega_\sigma(z)} \gamma(z)\, \Pi_{\mathbb{C}^k}.\end{aligned}$$

□

9.2 Verifying the jump formula conditions

Now that we have a contour integral representation for M_σ, the next step is to verify that we can apply the jump formula to move the contour. The jump formula requires that the integrand is in $\Theta(\mathbb{C}^k, \mathcal{P})$, where $\mathcal{P}$ is as in Proposition 2.

Proof of Proposition 2. By Theorem 2, $\mathcal{P}_1$ is locally finite and has finitely many normal spaces. The arrangement $\mathcal{P}_2$ is locally finite because a compact subset of $\mathbb{R}^k$ meets only finitely many coordinate hyperplanes $z_i = \beta$ for $\beta \in \mathbb{Z}_{\geq 0}$, and its normal spaces are spanned by subsets of the coordinate covectors. Hence $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ is locally finite. Its normal spaces are sums of a normal space from $\mathcal{P}_1$ and one from $\mathcal{P}_2$, so there are only finitely many of them.

By Definition 6,

$$\gamma(z) = \zeta(z) \prod_{i=1}^{k} \Gamma(-z_i).$$

By Theorem 9,

$$\zeta \in \widehat{R}(\mathbb{C}^k, \mathcal{P}_1) \subseteq \widehat{R}(\mathbb{C}^k, \mathcal{P}_1 \cup \mathcal{P}_2).$$

Meanwhile,

$$\prod_{i=1}^{k} \Gamma(-z_i) \in \widehat{R}(\mathbb{C}^k, \mathcal{P}_2) \subseteq \widehat{R}(\mathbb{C}^k, \mathcal{P}_1 \cup \mathcal{P}_2).$$

So $\gamma \in \widehat{R}(\mathbb{C}^k, \mathcal{P}_1 \cup \mathcal{P}_2)$ because $\widehat{R}(\mathbb{C}^k, \mathcal{P}_1 \cup \mathcal{P}_2)$ is an algebra. □

Corollary 9. *(→ proof) For every $\phi \in (\mathbb{R}^k)^*$,*

$$e^{-\phi}\gamma \in \Theta(\mathbb{C}^k, \mathcal{P}).$$

In particular,

$$\gamma \in \Theta(\mathbb{C}^k, \mathcal{P}) \qquad \text{and} \qquad e^{-\omega_\sigma}\gamma \in \Theta(\mathbb{C}^k, \mathcal{P}).$$

Moreover, for every chamber $\mathfrak{c} \in \Omega(\mathcal{P})$, γ restricted to $\mathfrak{c}+i\mathbb{R}^k$ satisfies the hypotheses of Proposition 15 with $V = \mathfrak{c}$.

Proof of Corollary 9. Let $\phi \in (\mathbb{R}^k)^*$. By Proposition 2, $\gamma \in \widehat{R}(\mathbb{C}^k, \mathcal{P})$. Since $e^{-\phi}$ is entire, $e^{-\phi}\gamma \in \widehat{R}(\mathbb{C}^k, \mathcal{P})$. So by Proposition 14, it is sufficient to show that for all $E \subseteq \mathbb{C}^k$ such that $\mathrm{Re}(E)$ is compact in the topology of $\mathbb{R}^k$ and $\mathrm{dist}(E, \mathcal{P}_\mathbb{C}) > 0$, there is $\delta > 0$ such that

$$\sup_{z \in E} (1 + |\mathrm{Im}(z)|)^{k+\delta} |e^{-\phi(z)}\gamma(z)| < \infty.$$

Let $E \subseteq \mathbb{C}^k$ be such that $\mathrm{Re}(E)$ is compact in the topology of $\mathbb{R}^k$ and $\mathrm{dist}(E, \mathcal{P}_\mathbb{C}) > 0$. By Corollary 3, there is $N \in \mathbb{Z}_{\geq 0}$ such that $|\zeta(z)| \ll (1 + |\mathrm{Im}(z)|)^N$ uniformly for $z \in E$. Also, $\sup_{z \in E} |e^{-\phi(z)}| < \infty$ because $|e^{-\phi(z)}| = e^{-\phi(\mathrm{Re}(z))}$ and $\mathrm{Re}(E)$ is compact. Meanwhile, by Stirling's formula for the Gamma function in vertical strips [29], $|\prod_{i=1}^{k} \Gamma(-z_i)|$ decays faster than any polynomial in $|\mathrm{Im}(z)|$, uniformly over compact real parts away from the polar set. The claim follows.

Taking $\phi = 0$ gives $\gamma \in \Theta(\mathbb{C}^k, \mathcal{P})$, and taking $\phi = \omega_\sigma$ gives $e^{-\omega_\sigma}\gamma \in \Theta(\mathbb{C}^k, \mathcal{P})$.

Let $\mathfrak{c} \in \Omega(\mathcal{P})$. If $c \in \mathfrak{c}$, then $c + i\mathbb{R}^k$ does not meet $\mathcal{P}_\mathbb{C}$, so γ is holomorphic on $\mathfrak{c} + i\mathbb{R}^k$. Hence $y \mapsto \gamma(c + iy)$ is measurable for every $c \in \mathfrak{c}$. Since $\gamma \in \Theta(\mathbb{C}^k, \mathcal{P})$, condition (2) of Proposition 15 holds. For arbitrary $\psi \in (\mathbb{R}^k)^*$, applying the local constancy condition in Definition 46 to $e^{-\psi}\gamma$ shows that

$$c \mapsto \int_{c+i\mathbb{R}^k} e^{-\psi}\gamma \, \Pi_{\mathbb{C}^k}$$

is locally constant on $\mathbb{R}^k \setminus \mathcal{P}$. Since each chamber $\mathfrak{c}$ is connected, this integral is constant on $\mathfrak{c}$. This gives condition (3) of Proposition 15. $\square$

9.3 Completion of the main theorems

We are finally ready to prove the main theorems.

Proof of main theorem, version 2. Let $\mathcal{P} \subsetneq \mathbb{R}^k$ be the locally finite real affine hyperplane arrangement of Proposition 2, and let $\mathfrak{c}_0 \in \Omega(\mathcal{P})$ be the chamber containing c_0.

By Proposition 1,

$$M_\sigma = \frac{1}{(2\pi i)^k} \int_{c_0 + i\mathbb{R}^k} e^{-\omega_\sigma(z)} \gamma(z)\, \Pi_{\mathbb{C}^k} = J_\sigma(\mathfrak{c}_0, \omega).$$

By Corollary 9, $e^{-\omega_\sigma}\gamma \in \Theta(\mathbb{C}^k, \mathcal{P})$. Applying Theorem 12 with $f = e^{-\omega_\sigma}\gamma$, we obtain, for all $L \in \mathbb{Z}_{\geq 1}$,

$$\begin{aligned} M_\sigma = &\sum_{j=0}^{L-1} \sum_{(\mathfrak{c}_0,\ldots,\mathfrak{c}_j) \in \mathbb{P}_j(\mathfrak{c}_0, G_\omega(\mathcal{P}))} \operatorname{sign}(\mathfrak{c}_0, \ldots, \mathfrak{c}_j) I_\sigma(\mathfrak{c}_j, \omega) \\ &+ \sum_{(\mathfrak{c}_0,\ldots,\mathfrak{c}_L) \in \mathbb{P}_L(\mathfrak{c}_0, G_\omega(\mathcal{P}))} \operatorname{sign}(\mathfrak{c}_0, \ldots, \mathfrak{c}_L) J_\sigma(\mathfrak{c}_L, \omega), \end{aligned}$$

where

$$I_\sigma(\mathfrak{c}, \omega) := \begin{cases} \dfrac{1}{(2\pi i)^{\dim F(\mathfrak{c},\omega)}} \displaystyle\int_{c + i \operatorname{lin}(F(\mathfrak{c},\omega))} & : \lambda(\mathfrak{c}, \omega) < \infty, \\ \quad \operatorname{Res}_{FL(\mathfrak{c},\omega)}(e^{-\omega_\sigma}\gamma\, \Pi_{\mathbb{C}^k}) & \quad c \in \operatorname{relint}(F(\mathfrak{c}, \omega)), \\ 0 & : \lambda(\mathfrak{c}, \omega) = \infty, \end{cases}$$

$$J_\sigma(\mathfrak{c}, \omega) := \frac{1}{(2\pi i)^k} \int_{c + i\mathbb{R}^k} e^{-\omega_\sigma(z)} \gamma(z)\, \Pi_{\mathbb{C}^k},$$
$$c \in \mathfrak{c},$$

and I_σ and J_σ are independent of c. We also record a conjugation symmetry which will show that the polynomials $p_{\mathfrak{c},\omega}$ are real-valued. On H^k,

$$\zeta(\overline{z}) = \int_W \prod_{i=1}^k f_i(w)^{\overline{z_i}} \phi(w)\, dw = \overline{\zeta(z)}.$$

By uniqueness of meromorphic continuation, the same identity holds for the meromorphic continuation of ζ for all $z \in \mathbb{C}^k$ away from the polar set. Since $\Gamma(\overline{w}) = \overline{\Gamma(w)}$, it follows that

$$\gamma(\overline{z}) = \overline{\gamma(z)}$$

away from the polar set. Hence $h_\sigma(z) := e^{-\omega_\sigma(z)}\gamma(z)$ satisfies

$$h_\sigma(\overline{z}) = \overline{h_\sigma(z)}$$

away from the polar set. If $c \in \mathbb{R}^k \setminus \mathcal{P}$, then parametrizing $c + i\mathbb{R}^k$ by $z = c + iy$ gives

$$\frac{1}{(2\pi i)^k} \int_{c+i\mathbb{R}^k} h_\sigma(z)\, \Pi_{\mathbb{C}^k} = \frac{1}{(2\pi)^k} \int_{\mathbb{R}^k} h_\sigma(c+iy)\, dy.$$

Taking complex conjugates and changing variables $y \mapsto -y$ shows that this contour integral is real. Thus, for every chamber $\mathfrak{c}$, $J_\sigma(\mathfrak{c}, \omega)$ is real. If $\lambda(\mathfrak{c}, \omega) < \infty$, the one-step identity obtained from the affine jump formula in the proof of Theorem 12 gives

$$I_\sigma(\mathfrak{c}, \omega) = J_\sigma(\mathfrak{c}, \omega) - \sum_{\alpha \in \{\pm 1\}^r \setminus \{1^r\}} \operatorname{sign}(\mathfrak{c}, \mathfrak{b}_{FL(\mathfrak{c},\omega)}(\alpha)) J_\sigma(\mathfrak{b}_{FL(\mathfrak{c},\omega)}(\alpha), \omega),$$

where $r = \operatorname{codim}(F(\mathfrak{c}, \omega))$. Therefore $I_\sigma(\mathfrak{c}, \omega) \in \mathbb{R}$ for every $\sigma > 0$. If $\lambda(\mathfrak{c}, \omega) = \infty$, this is immediate from $I_\sigma(\mathfrak{c}, \omega) = 0$.

Since $c_0 \in (-\infty, 0)^k$ and $\theta \in \mathbb{R}^k_{>0}$, we have $\omega(c_0) > 0$, so $\lambda(\mathfrak{c}_0, \omega) \geq \omega(c_0) > 0$. If $(\mathfrak{c}, \mathfrak{b})$ is an edge of $G_\omega(\mathcal{P})$ such that $\lambda(\mathfrak{c}, \omega) < \infty$, then Proposition 4 gives $F(\mathfrak{c}, \omega) \subsetneq \overline{\mathfrak{b}}$, hence $\lambda(\mathfrak{b}, \omega) \geq \lambda(\mathfrak{c}, \omega)$. In the case $\lambda(\mathfrak{c}, \omega) = \infty$, then $\lambda(\mathfrak{b}, \omega) = \lambda(\mathfrak{c}, \omega)$, so in either case $\lambda(\mathfrak{b}, \omega) \geq \lambda(\mathfrak{c}, \omega)$. Therefore every chamber appearing in a path from $\mathfrak{c}_0$ has $\lambda(\mathfrak{c}, \omega) \in \mathbb{R}_{>0} \cup \{\infty\}$.

Now fix $\eta \in \mathbb{R}$. By Proposition 6, there exists $L_0 \in \mathbb{Z}_{\geq 1}$ such that for all $L \geq L_0$ and all paths $(\mathfrak{c}_0, \ldots, \mathfrak{c}_L) \in \mathbb{P}_L(\mathfrak{c}_0, G_\omega(\mathcal{P}))$, we have $\lambda(\mathfrak{c}_L, \omega) > \eta$. By Corollary 9, γ satisfies the hypotheses of Proposition 15 on $V = \mathfrak{c}_L$. Since $\sup_{v \in \mathfrak{c}_L} \omega(v) = \lambda(\mathfrak{c}_L, \omega) > \eta$, Proposition 15 gives

$$J_\sigma(\mathfrak{c}_L, \omega) \in o(e^{-\sigma\eta}).$$

Next fix $\mathfrak{c} \in \Omega(\mathcal{P})$ such that $\lambda(\mathfrak{c}, \omega) < \infty$. By Proposition 8, Corollary 8, and Definition 30,

$$I_\sigma(\mathfrak{c}, \omega) = e^{-\sigma\lambda(\mathfrak{c},\omega)} p_{\mathfrak{c},\omega}(\sigma).$$

The conjugation argument above shows that $I_\sigma(\mathfrak{c}, \omega) \in \mathbb{R}$ for every $\sigma > 0$, and $e^{-\sigma\lambda(\mathfrak{c},\omega)} \in \mathbb{R}_{>0}$. Hence $p_{\mathfrak{c},\omega}(\sigma) \in \mathbb{R}$ for every $\sigma > 0$. Since $p_{\mathfrak{c},\omega}$ is real-valued on $(0, \infty)$, its Taylor coefficients at any positive point are real, so $p_{\mathfrak{c},\omega}$ has real coefficients. Since I_σ is independent of c, the polynomial $p_{\mathfrak{c},\omega}$ is also independent of $c \in \operatorname{relint}(F(\mathfrak{c}, \omega))$. □

Proof of Corollary 2. This is shown in the proof of Theorem 3. □

Proof of main theorem, version 1. Let $\mathcal{P}$ and $\mathfrak{c}_0$ be as in the main theorem, version 2. Set

$$\mathcal{I} := \{\mathfrak{c} \in \Omega(\mathcal{P}) : \mathfrak{c} \text{ lies on a path in } G_\omega(\mathcal{P}) \text{ starting at } \mathfrak{c}_0 \text{ and } \lambda(\mathfrak{c}, \omega) < \infty\}.$$

For each $\iota = \mathfrak{c} \in \mathcal{I}$, set $\lambda_\iota := \lambda(\mathfrak{c}, \omega)$ and $p_\iota := p_{\mathfrak{c},\omega}$. By Theorem 3, each $\lambda_\iota > 0$.

Given $\eta \in \mathbb{R}$, Proposition 6 gives $L_0 \in \mathbb{Z}_{\geq 1}$ such that for all $L \geq L_0$ and all paths $(\mathfrak{c}_0, \ldots, \mathfrak{c}_L) \in \mathbb{P}_L(\mathfrak{c}_0, G_\omega(\mathcal{P}))$, we have $\lambda(\mathfrak{c}_L, \omega) > \eta$. Proposition 15 then gives $J_\sigma(\mathfrak{c}_L, \omega) \in o(e^{-\sigma\eta})$ for all such terminal chambers. Fix such an L. Since each vertex of $G_\omega(\mathcal{P})$ has finitely many outgoing edges, $\mathbb{P}_j(\mathfrak{c}_0, G_\omega(\mathcal{P}))$ is finite for every $0 \leq j \leq L$. By Theorem 3,

$$\begin{aligned} M_\sigma = &\sum_{j=0}^{L-1} \sum_{(\mathfrak{c}_0,\ldots,\mathfrak{c}_j) \in \mathbb{P}_j(\mathfrak{c}_0, G_\omega(\mathcal{P}))} \operatorname{sign}(\mathfrak{c}_0, \ldots, \mathfrak{c}_j)\, I_\sigma(\mathfrak{c}_j, \omega) \\ &+ \sum_{(\mathfrak{c}_0,\ldots,\mathfrak{c}_L) \in \mathbb{P}_L(\mathfrak{c}_0, G_\omega(\mathcal{P}))} \operatorname{sign}(\mathfrak{c}_0, \ldots, \mathfrak{c}_L)\, J_\sigma(\mathfrak{c}_L, \omega). \end{aligned}$$

The second sum is a finite sum of $o(e^{-\sigma\eta})$ terms, hence is itself $o(e^{-\sigma\eta})$. For the first sum, define $\chi_\eta : \mathcal{I} \to \mathbb{Z}$ by

$$\chi_\eta(\mathfrak{c}) := \sum_{j=0}^{L-1} \sum_{\substack{(\mathfrak{c}_0,\dots,\mathfrak{c}_j)\in\mathbb{P}_j(\mathfrak{c}_0,G_\omega(\mathcal{P}))\\ \mathfrak{c}_j=\mathfrak{c}}} \operatorname{sign}(\mathfrak{c}_0,\dots,\mathfrak{c}_j).$$

Because $\mathbb{P}_j(\mathfrak{c}_0, G_\omega(\mathcal{P}))$ is finite for every $0 \le j$, only finitely many chambers appear in these paths, so χ_η has finite support. Since $I_\sigma(\mathfrak{c},\omega) = e^{-\sigma\lambda(\mathfrak{c},\omega)}p_{\mathfrak{c},\omega}(\sigma)$ if $\lambda(\mathfrak{c},\omega) < \infty$ and $I_\sigma(\mathfrak{c},\omega) = 0$ otherwise, collecting terms by chamber gives

$$M_\sigma = \sum_{\iota\in\mathcal{I}} \chi_\eta(\iota)\, e^{-\sigma\lambda_\iota}\, p_\iota(\sigma) + o(e^{-\sigma\eta}).$$

□

10 Conclusion

This thesis studied Laplace integrals associated to non-negative functions that are analytic up to smooth units, as well as the multivariate zeta function that encodes their local geometry. Motivated by questions from singular learning theory, the main objective was to understand how the polar structure of the multivariate zeta function determines the asymptotic behavior of the corresponding Laplace integral. The main theorem shows that

$$M_\sigma := \int_W \exp\Big(-\sum_{i=1}^{k} t_{\sigma i} f_i(w)\Big)\phi(w)\, dw$$

admits an asymptotic expansion. We show that for every $\eta \in \mathbb{R}$, M_σ can be written as a finite sum of terms of the form

$$p_\iota(\sigma)e^{-\sigma\lambda_\iota},$$

up to $o(e^{-\sigma\eta})$ error. Each $\lambda_\iota > 0$ is the optimal value of a linear program determined by the polar arrangement of the multivariate zeta function and p_ι is a polynomial with degree controlled by the polar order at the corresponding optimal face. The proof establishes the meromorphic continuation of the multivariate zeta function, develops a residue theory with a jump formula for crossing faces of the polar affine hyperplane arrangement, and gives a procedure to traverse the arrangement that reveals the dominant residues and controls the remainder up to any desired exponential rate.

From the perspective of singular learning theory, this thesis provides a framework for studying several analytic functions simultaneously, rather than a single loss function in isolation. Future work can leverage this framework to study the relationships between multiple datasets, the effects of regularization, and any other settings that fit into this general framework.

This thesis also isolates an interesting open problem that is not resolved here: it is not yet understood when the coefficient of the first formally available term in the expansion vanishes because of either cancellation in its complex contour integral representation or cancellation among multiple residue contributions. This question appears subtle and mathematically interesting in its own right. Along with the previously mentioned contributions, this work should therefore be viewed as identifying this precise open problem and providing machinery to address it in future work.

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